MODULE 3

Matrices and Linear Simultaneous Equations

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3.1 Introduction

Our last module, Module 7 of this course, is on numerical of linear simultaneous equations. An elegant mathematical format of representing such systems of linear equations is by using the concept of matrices. This module highlights the essential properties of 2×2 and 3×3 matrices and relates them to the formulation of systems of linear equations of the same order. In the sequel, we shall define matrices, operate with matrices, define the determinant and the inverse of a matrix and formulate systems of linear equations using matrices.

3.2 Objectives

At the end of this module the learner is expected to be able to

- Define a matrix
- Define a square matrix
- Define the order of a square matrix
- Define the zero and unit matrices
- Add, subtract and multiply same order matrices,
- Multiply a matrix by a scalar (real number),

- Calculate the determinant of a square matrix,
- Find the inverse of a nonsingular matrix, and
- Define a system of simultaneous linear equations.

3.3 Two-by-Two and Three-by-Three Matrices

Definition

A matrix is a **rectangular array of real numbers**, consisting of **rows** and **columns**, as given by the examples below.

Examples of Matrices

(i) $A = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$; (ii) $B = \begin{pmatrix} 3 & -2 & 1 \\ -1 & 3 & -3 \end{pmatrix}$; (iii) $C = \begin{pmatrix} -2 & 3 & -1 \\ 1 & -2 & 3 \\ -3 & 1 & 2 \end{pmatrix}$

Notation

Matrices are always denoted by capital letters. The individual numbers appearing in a matrix are called elements of the matrix. They are identified by their position in the matrix, indicating in which row and which column they appear. The element a_{ij} of a matrix is found in row number i counted from top to bottom and column number j counted from left to right. Thus the element a_{23} is found in the second row and third column of some matrix.

Type of a Matrix

The matrix in the first example has two rows and two columns. The second example has two rows and three columns and the third example has three rows and three columns. The type of a matrix shows how many rows and how many columns the matrix has and has the general form $r \times c$, where r is the number of rows, and c is the number of columns.

Accordingly, the examples given above are of types 2×2, 2×3 and 3×3, respectively.

Order of a Matrix

Matrices having the same number of rows and columns are called square matrices. We shall only deal with square matrices of order two and three in this module. This is because the type of systems of linear equations we shall solve in Module 7 are systems of two equations in two unknowns and three equations in three unknowns, respectively.

The Zero Matrix and the Unit Matrix

Two matrices merit special mention. These are the zero matrix and the unit matrix.

Definition

A zero matrix is any matrix all of whose elements are zero. Examples of zero matrices are:

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \quad Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Definition

A **unit matrix** or also referred to as the **identity matrix**, is a square matrix with unit entries along its top-left bottom-right diagonal (also called the main diagonal) and zero entries elsewhere. Examples of unit matrices are:

$$R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The **zero matrix** is simply denoted by O, plays a role similar to that of the number zero in adding and subtracting numbers.

$A \pm O = A; \qquad A - A = O; \qquad AO = OA = O$

Equally true, the unit matrix, denoted by I, plays a role similar to the number 1 in multiplication: $IA = AI$

3.4 Matrix Operations

Although matrices are made of numbers, matrices are not numbers and therefore there is need to define some of the arithmetic operations are compatible with matrices.

The arithmetic operations we can perform with (square) matrices are:

(i) Product of a matrix and real number (multiplication by a scalar)
(ii) Sum and Difference of two matrices (Addition and subtraction)
(iii) Product of two matrices (Multiplication of two matrices

We illustrate these operations on some general 3×3 matrices

We define three general 3×3 matrices as follows:

Scalar multiplication

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}; \quad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}; \quad C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}.$$

If k is any real number, the $kA = \begin{pmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{pmatrix}$, obtained by

multiplying each element of the matrix A by the number k.

Addition and subtraction

If $A+B=C$, then $c_{ij}=a_{ij}+b_{ij}$

If $A-B=C$, then $c_{ij}=a_{ij}-b_{ij}$

Matrix Product

If $AB=C$, then $c_{ij}=R_i(A)\times C_j(B)=\sum_{k=1}^{3}a_{ik}b_{kj}$ where $R_i(A)$ denotes the $i-th$ row of matrix A and $C_j(B)$ denotes the $j-th$ column of matrix B.

As typical examples of elements of the product observe the following:

$$c_{11}=a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}\,; \quad c_{23}=a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33}$$

3.5 Determinant of a Matrix

The determinant of a matrix is defined only for square matrices. The determinant of a square matrix is a real number. If the determinant is not zero, then we shall soon learn in the following section that the determinant is needed in the process of finding the inverse of a matrix.

Notation

The determinant of a square matrix A is denoted either by the symbol $|A|$ or simply by writing $\det(A)$.

Definition

(i) If $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then $\det(A)=|A|=a_{11}a_{22}-a_{21}a_{22}$.

The determinant of matrix $A=\begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$ is $(2)(5)-(-1)(3)=13$.

(ii) If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

then, $\det(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{32} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$.

The determinant of matrix $C = \begin{pmatrix} -2 & 3 & -1 \\ 1 & -2 & 3 \\ -3 & 1 & 2 \end{pmatrix}$

is $-2\begin{vmatrix} -2 & 3 \\ 1 & 2 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ -3 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 1 & -2 \\ -3 & 1 \end{vmatrix} =$

$$-2(-4-3) - 3(2-3) - 1(1-6) = 14 - 33 + 5 = -14.$$

Singular Matrices

A singular matrix is a square matrix whose determinant is zero. If the determinant of a square matrix is different from zero, then the matrix is said to be nonsingular.

The matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ is singular, but matrix $B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ is nonsingular.

3.6 Inverse Matrix

The inverse of a square matrix is a matrix A is a matrix B such that the product $AB = BA = I$. In order to bring out more clearly this special relationship between the two matrices A and B one uses the familiar notation $B = A^{-1}$ so that $A^{-1}A = AA^{-1} = I$. As we shall learn in Module 7 the inverse matrix has a theoretical significance in solving systems of simultaneous linear equations.

3.7 Linear Simultaneous Equations

Systems of simultaneous linear equations are frequently encountered in solving a number of mathematical problems. One such problem is the least squares method of finding lines of best fit in approximation theory.

We shall only deal with systems having two equations in two unknowns, and three equations in three unknowns.

The general 2×2 and 3×3 lsystems of simultaneous linear equations areas given below:

(i)
$$\begin{array}{ccccc} a_{11}x_1 & + & a_{12}x_2 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & = & b_2 \end{array},\quad \text{and}$$

(ii)
$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & = & b_3 \end{array}$$

where all the coefficients $a_{ij}'s$ are given (known) numbers and one seeks to find values of the x_i,s such that all two or all three equations are satisfied at the same time (simultaneously).

These systems can also be reformulated using matrix-vector notation as follows.

(i) $$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix};$$ (ii) $$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{33} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

3.8 Summary

In this module we have highlighted all we need in preparation for deriving and applying numerical methods for solving systems of simultaneous linear equations. An analytical (exact) solution of such problems can be got if the inverse of the matrix of coefficients is easy to find. Crammer's method requires the calculation of lots of determinants. These will be found to be of little practical use because of their inefficiency (too many and complicated computations). This will justify the use of numerical methods which, although approximate, are capable of giving good approximate solutions relatively efficiently.