MODULE 5

ROOTS OF FUNCTIONS

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5.1 Introduction

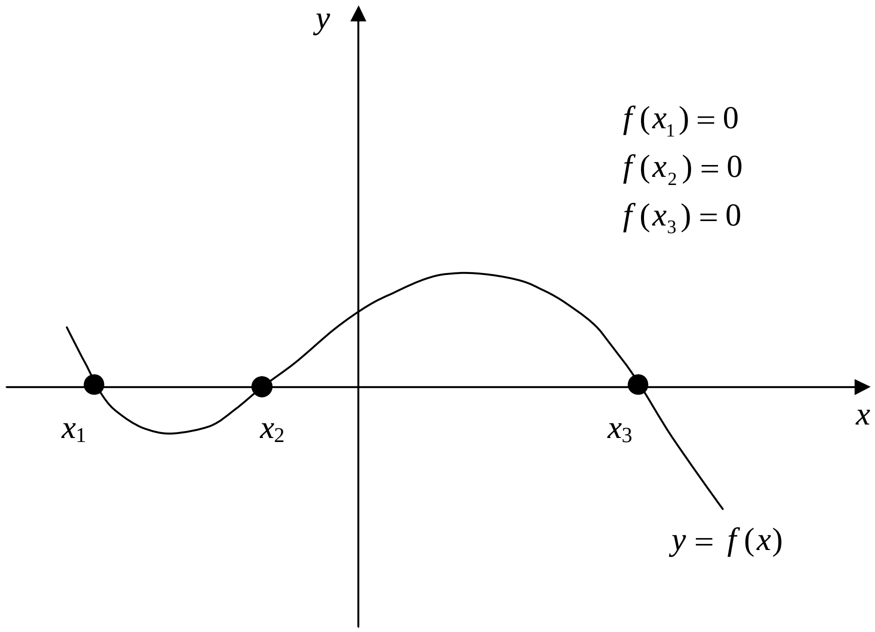
Four numerical methods for finding roots of functions will be presented in this module. All four methods are for solving the nonlinear equation $f(x) = 0$.

The bisection method will be derived. We shall prove that the method is always convergent. This will be followed by the method known as regula-falsi, which has significant similarity with the bisection method but converges slightly faster.

The Secant method is immediately presented after the regula-falsi method because the two methods share a common mathematical formula. However, the secant method is computationally more efficient. The Newton-Raphson method is presented last and is shown to be a generalized form of the secant and regula-falsi methods.

Definition of a Root or a Zero of a Function

For a function of a single independent variable $y = f(x)$, a point $x = \rho$ is called a root or a zero of $f(x)$ if the value of the function is zero at the point, meaning $f(\rho)=0$. In the following figure, the points x_1, x_2, x_3 are all roots of the function $f(x)$.



5.2 Objectives

On completion of this module the learner is expected to be able to

- **Derive and apply** the bisection method
- **Prove** the convergence of the bisection method
- **Derive and apply** both the Secant method and the Regula-Falsi method
- **Derive and apply** the Newton-Raphson method

5.3 Derivation and application of the bisection method

The bisection method for approximating roots of functions is a typical example of an iteration method.

In its simplest form, an iteration method may be defined as a repetitive process of applying a function $g(x)$ on one or several previous approximate values x_{n-1} ... to produce a new and hopefully more accurate approximation x_n of the specific quantity being sought. In mathematical symbolism we write

$$\boxed{x_n = g(x_{n-1}, x_{n-2}, x_{n-3}, \ldots)}.$$

The bisection method is based on a special interpretation of the intermediate value theorem for continuous functions. This states that, if the function $f(x)$ is continuous on an interval $[a,b]$ and if the values $f(a)$ and $f(b)$ differ in sign, $f(a)f(b) < 0$, then the equation $f(x) = 0$ has at least one real root $\rho \in [a,b]$.

Assuming that the points a and b were chosen to contain only one root we can bisect the interval $[a,b]$ into two halves at the point $c = \frac{1}{2}(a+b)$ and conclude that the root lies either in the interval (a,c) or in the interval (c,b) provided that $f(c) \neq 0$, in which case c is the required root.

The Bisection method repeats the process of bisecting the interval which contains the root ρ until we are satisfied that we close enough to the root.

The above process can be summed up by the following algorithm (Kendal E. Atkinson, 1989 pp 56).

Algorithm "Bisect ($f(x)$, a, b, ρ, ε **)"**

Steps:

1. Set $x_1 := a$ and $x_2 := b$
2. Define $x_3 = \frac{1}{2}[x_1 + x_2]$
3. If $x_2 - x_3 \leq \varepsilon$, then accept $\rho = x_3$ and exit.
4. If $f(x_2)f(x_3) \leq 0$, then $x_1 := x_3$; otherwise $x_2 := x_3$.
5. Go back to step 2.

Application of the bisection method

Verify that the function $f(x) = x^2 + 4x - 10$ has a root inside the interval $(1,2)$ and use the limits of the interval as starting values of the bisection method to approximate the root in 10 bisections.

Solution
We evaluate the function at the two end points of the given interval and find $f(1) = -5$ and $f(2) = 2$. Since $f(x)$ is a continuous function and $f(1)f(2) < 0$, the intermediate value theorem asserts that f has at least one root in the interval $1 \le x \le 2$. We can therefore carry out the bisection method to find the results tabulated hereunder.

Table 4.1: Bisection Method for the function $f(x) = x^2 + 4x - 10$

n	a	$f(a)$	b	$f(b)$	c	$f(c)$
	1	-5	2	2	1.5	-1.75
1	1.5	-1.75	2	1.5	1.75	0.0625
2	1.5	-1.75	1.75	0.0625	1.625	-0.859375
3	1.625	-0.859375	1.75	0.0625	1.6875	-0.402344
4	1.6875	-0.402344	1.75	0.0625	1.71875	-0.170898
5	1.71875	-.170898	1.75	0.0625	1.734375	-0.054443
6	1.734375	-0.054443	1.75	0.0625	1.742188	0.003971
7	1.734375	-0.054443	1.742188	0.003971	1.738282	-0.025248
8	1.738282	-0.025248	1.742188	0.003971	1.740235	-0.010642
9	1.740235	-0.010642	1.742188	0.003971	1.741212	-0.003333
10	1.741212	-0.003333	1.742188	0.003971	1.741700	0.000319

The sequence of values under the c-column in the table is convergent. The true value of the root being approached by this sequence (use the quadratic formula) is $\rho = 1.741657...$

5.4 Proof of convergence of the bisection method

Convergence of any iteration method implies that the error in the approximation is tending to zero as the number of iteration increases.

For the Bisection method, the absolute value of the error is bounded by the length of the interval in which the root lies at that particular stage. Without loss of generality, we assume that $a < b$ so that $b - a > 0$.

Error bound after 1st bisection $\quad |\rho - x_3| = \varepsilon_1 \le \frac{1}{2}[b - a]$.

Error after 2nd bisection $|\rho - x_4| = \varepsilon_2 \le \frac{1}{2}\left(\frac{b-a}{2}\right) = \frac{b-a}{2^2}$

Error after 3rd bisection $|\rho - x_5| = \varepsilon_3 \le \frac{1}{2}\left(\frac{b-a}{2^2}\right) = \frac{b-a}{2^3}$

Error after n^{th} bisection $|\rho - x_6| = \varepsilon_n \le \frac{1}{2}\left(\frac{b-a}{2^{n-1}}\right) = \frac{b-a}{2^n}$

Because $\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty}\left(\frac{b-a}{2^n}\right) = 0$ we conclude that the bisection method always converges.

5.5 Derivation and application of the secant and the regula-falsi methods

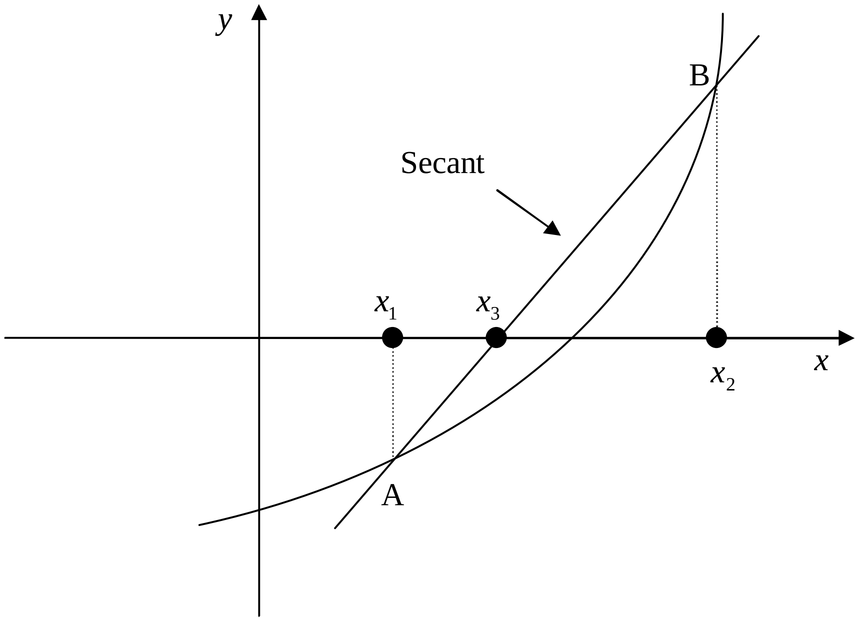
The Bisection method we have just presented was a bit wasteful. In it one spends great efforts at calculating values of the function f at two points which are only used for deciding in which subinterval the root lies but are never used in the actual calculation of the approximate value of the root.

The method known as Regula Falsi (or rule of false position) rectifies this anomaly. The method retains the **principle of enclosure** characteristic of the Bisection method but also makes direct use of the function values at the two points which enclose the root of the function.

The general approach for this and subsequent methods (Secant method and the Newton Raphson method) is to replace the curve of $y = f(x)$ in the interval where the root lies by the straight line joining the two points.

Specifically, let the root ρ lie in the interval $[x_1, x_2]$. The equation of the secant line joining the two points $A(x_1, f_1)$ and $B(x_2, f_2)$ is represented by the equation

$$\boxed{y = f_1 + \frac{f_2 - f_1}{x_2 - x_1}(x - x_1)}\,.$$



This line intersects the x-axis at the point with coordinates $(x_3, 0)$ where

$$x_3 = x_1 - \left[\frac{x_2 - x_1}{f_2 - f_1}\right] f_1 = \frac{x_1 f_2 - x_2 f_1}{f_2 - f_1}$$

Similar to the bisection method, the regula-falsi also assumes that the two values x_1 and x_2 enclose the root being sought. It therefore requires that $f_1 f_2 < 0$ and at the same time involves the same function values in calculating a new approximation to the root.

The above process can be repeated several times in an iterative process using the following algorithm.

Algorithm "Regula Falsi ($f(x), \quad a, \quad b, \quad \rho, \quad \varepsilon$ **)"**

Steps:

1. Define $x_3 = x_1 - \left[\frac{x_2 - x_1}{f_2 - f_1}\right] f_1$.
2. If $|x_1 - x_3| \le \varepsilon$ and $|x_2 - x_3| \le \varepsilon$ then accept $\rho = x_3$ and exit.
3. If $f_2 f_3 \le 0$ then $x_1 := x_3$; otherwise $x_2 := x_3$.
4. Go back to step 1.

Example

Starting with the values $x_1 = 1, x_2 = 2$ apply the regula-falsi method on the function $f(x) = x^2 + 4x - 10$ to obtain an approximate value of the root enclosed in the interval (x_1, x_2) in only four (4) iterations.

The Regula Falsi Method for the function $f(x) = x^2 + 4x - 10$

n	x_1	$f(x_1)$	x_2	$f(x_2)$	$x_3 = \dfrac{x_1 f_2 - x_2 f_1}{f_2 - f_1}$	$f(x_3)$
	1	-5	2	2	1.714286	-0.181785
1	1.714286	-0.181785	2	2	1.740842	-0.006101
2	1.740842	-0.006101	2	2	1.741630	-0.000205
3	1.741630	-0.000205	2	2	1.741656	-0.000010
4	1.741656	-0.000010	2	2	1.741657	-0.000003

Examination of the values of x_3 against the background of the exact value of the root $\rho = 1.741657387...$ demonstrates the fact that convergence of the regula-falsi method is faster compared to the bisection method.

Observation

The requirement by both the bisection method and the regula-falsi methods that the two values involved in the calculations, x_1, x_2 must enclose the root is computationally very restrictive and significantly reduce the efficiency of both methods. This is because in programming the two methods, the process of checking whether or not $f(x_1)f(x_2) < 0$ is time consuming. As a result, efforts have been made to do without this check. Our next method achieves just that.

Derivation of the Secant Method

The Secant method is essentially the same as the regula-alsi method. The only difference is that in the Secant method requirement that the two values x_1 and x_2 must enclose the root is dropped. All one needs is to require the two values used in the computation be close enough to the required root. The algorithm for the Secant method is as given hereunder.

Algorithm "Secant Method ($f(x), \quad a, \quad b, \quad \rho, \quad \varepsilon$ **)"**

Steps:

1. Define $x_3 = x_1 - \left[\dfrac{x_2 - x_1}{f_2 - f_1}\right] f_1$.

5. If $|x_1 - x_3| \le \varepsilon$ and $|x_2 - x_3| \le \varepsilon$ then accept $\rho = x_3$ and exit.

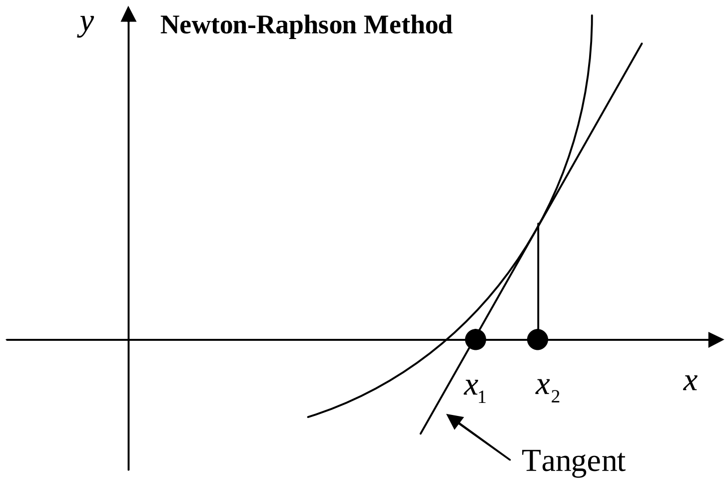
6. Otherwise set $x_1 := x_2$ and $x_2 := x_3$

7. Go back to step 1.

5.6 Derivation and Application of the Newton-Raphson Method

The Newton Raphson method is by far the most popular numerical method for approximating roots of functions. The method assumes that the function $f(x)$ is differentiable in the neighborhood of the root and that the derivative is not zero in anywhere in that neighborhood.

Assuming that x_0 is a point which is sufficiently close to the root of the function, the graph of the function $y = f(x)$ is approximated by the tangent to the curve at the point.



The equation of the tangent through point $(x_0, f(x_0))$ on the curve $y = f(x)$ is

$$\boxed{y = f_0 + (x - x_0) f^{/}(x_0)}$$

This tangent intersects with the x- axis at the point x_1 whose value is

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f^{/}(x_0)}}$$

The value x_1 is then accepted as a new approximation to the root. The point $(x_1, f(x_1))$ can be taken as a new point to draw a tangent through. Its intersection with the x- axis, given by $x_2 = x_1 - \frac{f(x_1)}{f^{/}(x_1)}$ is also accepted as a new approximation to the root. This process can be repeated over and over, leading to the iteration method given by the

Newton-Raphson formula $$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f^{/}(x_n)}, \qquad n = 0,1,2,\ldots}$$

Each iteration using the Newton Raphson method requires one function evaluation and one first derive evaluation. Compared to the previous three numerical methods, the Newton Raphson method converges very rapidly to the root.

Example

Starting with $x_0 = 1$ approximate a root of the function $f(x) = x^2 + 4x - 10$ correct to 6 decimal places.

Solution

$$f(x) = x^2 + 4x - 10 \quad = \quad (x+4)x - 10$$
$$f^{/}(x) = 2x + 4$$

The requirement that we obtain a answer correct to 6 decimal places simply means that we carry out the iteration until the 7th decimal digit in the values being calculated is no longer changing.

Application of the Newton-Raphson Method on the function
$f(x) = x^2 + 4x - 10$

n	x_n	$f(x_n)$ $(x_n+4)x_n-10$	$f^{/}(x_n)$ $2x_n+4$	$x_{n+1} = x_n - \frac{f(x_n)}{f^{/}(x_n)}$
0	1	-5	6	1.833 333 3
1	1.833 333 3	0.6944442	7.6666666	1.742 753 6
2	1.742 753 6	0.0082045	7.4855072	1.741 657 5
3	1.741 657 5	0	7.483315	1.741 657 5

Because the value of the function at x_3 is for all purposes zero, we can conclude that the required approximation is $x = 1.741657$ rounded to six decimal places. The true value of the root to the same degree of accuracy is

5.7 Summary

Except for some very special functions $f(x)$, the mathematical problem of finding roots of a nonlinear function $f(x)$ is a daunting task. Analytical methods are seldom available and hence the need for numerical methods.

We have derived four numerical methods which are relatively simple to derive and apply. Mastery of the methods largely depends on their usage. All four are not direct methods but iterative. They all converge provided the starting value(s) are sufficiently close to the root being sought. The bisection method is the slowest while the Newton-Raphson method is the fastest.

In solving any given problem the learner is strongly advised to work with at least six decimal places accuracy in order to be able to monitor the accuracy of the approximations manifested by the number of digits which remain constant during the iteration process.