MODULE 7

NUMERICAL SOLUTION OF LINEAR SIMULTANEOUS EQUATIONS

Module Content

- Introduction
- Objectives
- Analytical methods for solving systems of simultaneous linear equations
- Need for numerical methods
- Gaussian elimination
- Partial pivoting
- Summary

7.1 Introduction

This is the last module of our course OIT 102. The module addresses our fourth and last mathematical problem we wish to learn how to solve it numerically. Only one numerical method will be presented, the programmable Gaussian elimination. We emphasize the adjective **"programmable"** because this is different from the Gaussian elimination method learnt at lower levels of education on the visual aid of physically looking at individual equations in a given system on the basis of the size of their coefficients decide which row should be used to eliminate some variable.

7.2 Objectives

On completion of this module the learner is expected to be able to

- Give two analytical methods for solving a given system of simultaneous linear equations.

- Explain why the two analytical methods listed are not practical for large systems of linear equations
- Describe and derive the programmable Gaussian elimination method
- Apply the programmable Gaussian elimination method on a system of linear system of not more than order three
- Explain the meaning of and apply the technique of partial pivoting in conjunction with the programmable Gaussian elimination method.

7.3 Analytical Methods for Solving Systems of Simultaneous Linear Equations

In Module 3 we formulated the mathematical problem of systems of simultaneous linear equations. We reproduce here the two possible formulations, purely algebraic and using the matrix-vector formulation.

(i) A system consisting of two equations in two unknowns can be formulated either in the algebraic form

$$\begin{aligned} a_{11}x_1 &+ a_{12}x_2 = b_1 \\ a_{21}x_1 &+ a_{22}x_2 = b_2 \end{aligned}$$

or in the matrix-vector form

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

(ii) A system consisting of three equations in three unknowns can also be formulated either in the algebraic form

$$\begin{aligned} a_{11}x_1 &+ a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 &+ a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 &+ a_{32}x_2 + a_{33}x_3 = b_3 \end{aligned}$$

or in the matrix vector form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{33} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

The general 2×2 and 3×3 systems of simultaneous linear equations areas given below:

(i) $\begin{matrix} a_{11}x_1 & + & a_{12}x_2 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & = & b_2 \end{matrix}$, and

(ii) $\begin{matrix} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & = & b_3 \end{matrix}$

Important Observation

A square system of linear equations has a solution only if the matrix A of coefficients is **nonsingular**. This statement is equivalent to saying that $|A| \equiv \det(A) \neq 0$, implying that A^{-1} exists.

Two analytic methods are available for solving any system of linear equations for which A^{-1} exists.

(i) **Use of the inverse matrix A^{-1}.**

Assuming the system is in the matrix-vector form $A\underline{X} = \underline{b}$, where $\underline{X}$ is the vector of unknowns and $\underline{b}$ is the vector of known right hand side elements, the analytical solution $\underline{X}$ is calculated using the equation

$$\underline{X} = A^{-1}\underline{b}$$

(ii) **Crammer's Rule**

Crammer's rule is an analytical method which calculates each component of the solution using the formula

$$x_i = \frac{D_i}{D},\ i = 1,2,3,\cdots$$

where $D = \det(A)$ and D_i is the determinant of a matrix A_i obtained from the matrix A by replacing its $i-th$ column with the right-hand side vector $\underline{b}$.

For purposes of demonstration, we consider the following system of two equations in two unknowns.

$$\begin{array}{rcrcr} -3x_1 & + & 5x_2 & = & 7 \\ x_1 & - & 2x_2 & = & 3 \end{array}$$

or expressed in matrix-vector form $\begin{pmatrix} -3 & 5 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$.

Solution using the inverse matrix

One can verify that the inverse of the matrix of coefficients is $A^{-1} = \begin{pmatrix} -2 & -5 \\ -1 & -3 \end{pmatrix}$ and therefore we can apply the analytical method which uses the inverse matrix and get

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 & -5 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} 7 \\ 3 \end{pmatrix} = \begin{pmatrix} -29 \\ -16 \end{pmatrix}.$$

Solution using Crammer's Rule

One calculates three determinants, namely

$$D = \begin{vmatrix} -3 & 5 \\ 1 & -2 \end{vmatrix} = 1, \quad D_1 = \begin{vmatrix} 7 & 5 \\ 3 & -2 \end{vmatrix} = -29, \quad D_2 = \begin{vmatrix} -3 & 7 \\ 1 & 3 \end{vmatrix} = -16.$$

Crammer's rule then gives the solution $x_1 = \dfrac{D_1}{D} = -29$, $x_2 = \dfrac{D_2}{D} = -16$.

7.4 Nee for Numerical Methods

The two numerical methods discussed in the preceding Section 7.3 look quite simple and straight forward if viewed uncritically. One only needs to find the inverse matrix to apply the first method, and for the second method knowledge of how to calculate determinants is all one needs to know.

Things are not all that simple as they sound. Finding the inverse of a matrix is not a simple task. It involves calculating many determinants. Similarly for Crammer's rule. One has to calculate the determinant of the coefficient matrix, and as many other determinants as there are unknowns. For systems with a large number of unknowns ($n > 3$) evaluation of determinants is very tasking.

For the above reason, numerical methods are very much needed.

7.5 The Programmable Gaussian Elimination Method

We shall derive the method using the case of a linear system with three unknowns

$$\begin{array}{lllllllll} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & = & b_1 & & (E_1) \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & = & b_2 & & (E_2) \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & = & b_3 & & (E_3) \end{array}$$

The Gaussian elimination method aims at systematically reducing the given system to the **upper triangular form**:

$$\boxed{\begin{array}{lllllllll} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & = & b_1 & & (E_1) \\ & & a_{22}^{*}x_2 & + & a_{23}^{*}x_3 & = & b_2^{*} & & (E_2^{*}) \\ & & & & a_{33}^{**}x_3 & = & b_3^{**} & & (E_3^{**}) \end{array}}$$

This is done as follows:

Assume that $a_{11} \neq 0$.
We then use the first equation (E_1) to eliminate the unknown x_1 from the second and third equations. To achieve this we calculate the two multipliers $M_{21} = -\dfrac{a_{21}}{a_{11}}$ and $M_{31} = -\dfrac{a_{31}}{a_{11}}$ and form a new second equation (E_2^*) and a new third equation (E_3^*) as follows:

$$M_{21}(E_1) + (E_2): \qquad a_{22}^* x_2 \; + \; a_{23}^* x_3 \; = \; b_2^* \qquad (E_2^*)$$

$$M_{31}(E_1) + (E_3): \qquad a_{32}^* x_2 \; + \; a_{33}^* x_3 \; = \; b_3^* \qquad (E_3^*)$$

The coefficient of x_1 will be zero in both equations.

We now repeat the same process by considering the two new equations and using equation (E_2^*) to eliminate the unknown x_2 from equation (E_3^*).

We assume that $a_{22}^* \neq 0$, calculate the multiplier $M_{32} = -\dfrac{a_{32}^*}{a_{22}^*}$ and use to form a new third equation (E_3^{**}) from the relation

$$M_{32}(E_2^*) + (E_3^*): \qquad a_{33}^{**} x_3 \; = \; b_3^{**} \qquad (E_3^{**})$$

This coefficient of x_2 in this last equation is zero. The three equations (E_1), (E_2^*) and (E_3^{**}) constitute the upper triangular frame given above.

An upper triangular system is easily solved by backward substitution, using the last equation to calculate the value of x_3; substituting this value into the second (middle) equation and solving for x_2, and finally substituting x_2 and x_3 into the first equation to get the value of x_1 .

Worked Example

7.6 Summary