# MODULE 2

## *FUNCTIONS OF ONE INDEPENDENT VARIABLE*

### Module Content

- Introduction
- Objectives
- Continuous variable
- Limit of a function
- Continuity of a function
- Derivative of a function
- The indefinite integral of a function
- The definite integral of a function
- Summary

### 2.1 Introduction

The first three mathematical problems that we shall strive to solve numerical in this course involve functions of one independent variable. It is therefore important to have a module that focuses on the concept of a function of one variable.

### 2.2 Objectives

At the end of this module the learner will be able to:

- Understand the difference between discrete and continuous variables
- Define the concept of a function of one independent variable
- Define the concept of limit of a function and evaluate one-sided limits
- Define the concept of and determine continuity of a function at a point
- Define the derivative of a function and be able to differentiate different functions
- Define an indefinite and the definite integral of a function

## 2.3 Continuous Variables

## 2.4 Limit of a function

The concept of limit is fundamental to and characterizes the field of mathematical analysis generally referred to as calculus.

A function $f$ is said to have limit $L$ as $x$ approaches point $c$ if the values of $f(x)$ approach $L$ as $x$ approaches c from whichever side the point c is approached. We write $\lim_{x \to c} f(x) = L$.

**One sided limits**

A function $f$ may have different limits depending on the direction the point $c$ is approached. The limit obtained by approaching $c$ from the left (values less than $c$) is called the left-handed limit. The limit obtained by approaching $c$ from the right (values greater than $c$) is called the right-handed limit. We denote such limits by $\lim_{x \to c^+} f(x) = L^+$, and

$$\lim_{x \to c^-} f(x) = L^-$$

**Determination of Limits**

Consider the **one-sided limits** of the function $y = \frac{x^2 - 4}{x - 2}$ as $x$ approaches 2. Does the function have a limit at 2?

Note that the function is not defined at 2. However, this should not bother us much because ***"tending to 2"*** does not require being at the point itself. Indeed, this is an important observation to make regarding limits. The function need not be defined at the point.

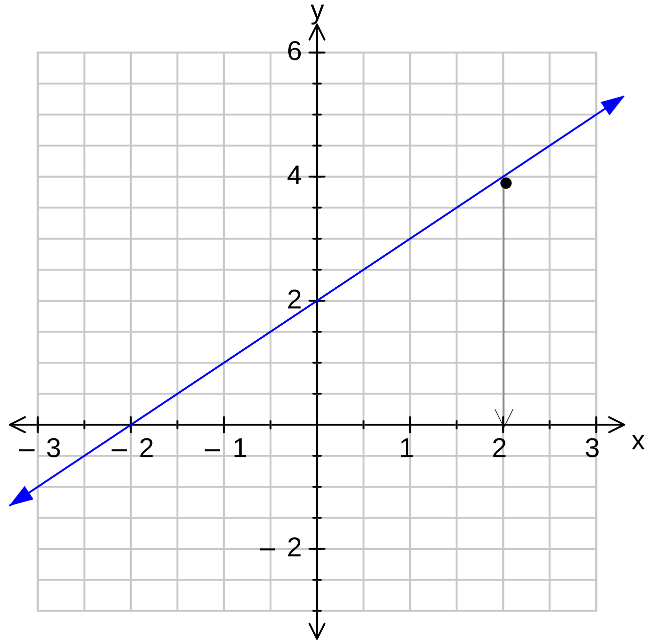
In order to solve the problem let us investigate the behavior of the function as x approaches 2 by evaluating it at values close to 2 on both sides of 2.

| $x$ | 1.9 | 1.95 | 1.97 | 1.99 | 2.0 | 2.01 | 2.03 | 2.05 | 2.1 |
|---|---|---|---|---|---|---|---|---|---|
| $y = f(x)$ | 3.9 | 3.95 | 3.97 | 3.99 | | 4.01 | 4.03 | 4.05 | 4.1 |

**DO THIS**

Draw the graph of the function.
How do you indicate on the graph that the number 2 is not in the domain of the function?

From the values given in the table, it is quite obvious that, as $x$ approaches 2 from either side, the function values approach 4. In this case the **left-hand** and **right-hand** limits are equal.




$$y = \frac{x^2 - 4}{x - 2}$$

## 2.5 Continuity of a function

A function $f$ is said to be continuous at a point $x = c$ if $\lim_{x \to c} f(x) = f(c)$.

This definition is equivalent to the following three conditions:

- The function is defined at the point, meaning that $f(c)$ exists,
- The function has a limit as $x$ approaches $c$, and
- The limit is equal to the value of the function.

### Continuity over an Interval

A function $f$ is continuous over an interval $I = [a,b]$ if it is continuous at each interior point of $I$ and is continuous on the right hand at $x = a$ and on the left hand at $x = b$.

### Discontinuity

If a function is not continuous at a point $c$ then one says the function is discontinuous at $x = c$. For example, $f(x) = \frac{\sin x}{x}$ is not continuous at $x = 0$.

The function $f(x) = \frac{x^2 - 4}{x - 2}$ whose limit was found to be 4 is also discontinuous at the point $x = 2$. It is discontinuous because it is not defined at the point. However, we can define a new function $F(x)$ using the function $f(x)$ which is continuous at $x = 2$ as follows: $F(x) = \begin{cases} f(x) & if \quad x \neq 2 \\ 4 & if \quad x = 2 \end{cases}$

$F(x)$ is continuous at $x = 2$ because $\lim_{x \to 2} F(x) = \lim_{x \to 2} f(x) = 4$ and $F(2) = 4$ so that $\lim_{x \to 2} F(x) = 4$. What we have done is to remove the discontinuity of $f(x)$ at $x = 2$ by assigning the value of $\lim_{x \to 2} f(x) = 4$ to be the value of $F(2)$.

### Removable discontinuity

A function $f(x)$ has a removable discontinuity at $x = c$ if $\lim_{x \to c} f(x) = L$ and either $f(c)$ does not exist or $f(c) \neq L$.

## 2.6 Derivative of a function

The derivative of a function $y = f(x)$ plays an important role in mathematics. Geometrically, the derivative of a function at a point is a measure of the slope of the curve representing the function at the point.

The derivative of a function $y = f(x)$ is defined either as $\lim_{x \to c}\left[\frac{f(x) - f(c)}{x - c}\right]$

or as $\lim_{h \to 0}\left[\frac{f(c+h) - f(c)}{h}\right]$ provided that the limit exists.

A function $f(x)$ is said to be differentiable at x = c if the above limits exists.

## 2.7 Indefinite and Definite Integral of a function

In the above section (1.6) you saw how differentiation of a function $f(x)$ led to a new function $f^{'}(x)$ called the derivative of $f(x)$. Now we discuss another possibility of deriving a function $F(x)$ from a given function $f(x)$. We denote the new function by $F(x) = \int f(x)dx$.

This function is referred to as an **indefinite integral**, **anti-derivative** or a **primitive of** $f$ **.**

An expression for the function $F(x)$ is obtained by requiring that its derivative is $f(x)$.

$$F^{'}(x) = f(x).$$

The process of finding an anti-derivative is referred to as **anti-differentiation** or integration. It is the inverse of differentiation.

In the current activity you will learn rules and apply various methods of integration on a variety of functions, including polynomial, exponential, logarithmic and trigonometric functions.

## Key Concepts

### Anti-derivatives

Anti-derivatives are is a family of functions $\{F(x) + c\}$ where $c$ is an arbitrary constant of integration, with a common derivative $f(x)$.

### Anti-differentiation

Anti-differentiation is the process of going from a derivative function $f(x)$ to a function $F(x)$ that has that derivative.

**Indefinite integral**
An anti-derivative is also referred to as an indefinite integral due to the presence of the arbitrary constant of integration.

**Primitive function**
If $f$ is a function defined on an interval $I$, then the function $F$ is a primitive of $f$ on $I$ if $F$ is differentiable on $I$ and $F'(x) = f(x)$.

## Key Theorems and/or Principles

**Rules for Integration**

(i) $\int af(x)dx = a\int f(x)dx$

(ii) $\int\left(\sum_{i=1}^{n} a_i f_i(x)\right)dx = \sum_{i=1}^{n} a_i\left(\int f_i(x)dx\right)$

(iii) $\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$

(iv) $\int x^{-1}dx = \ln|x| + C$

There are two ways of introducing the concept of integration. Historically the concept of integration was introduced via the problem of finding the area below a curve lying between two given ordinates. This is the Riemann approach that leads to a sequence of Riemann Sums that under some fairly simple assumptions on the function $f$ can be shown to converge to the number (area).

The other way of introducing the integral of a function is by viewing integration as an inverse operation to differentiation. This led to calling the process **anti-differentiation** and the outcome of the process being called an **anti-derivative** or a **primitive** of the function $f$.

**Definition**
A function $F(x)$ is called a primitive (or anti-derivative) of another function $f(x)$ if the derivative of $F(x)$ is $f(x)$:

$$F'(x) = f(x).$$

One denotes $F(x)$ by $F(x) = \int f(x)dx$. The symbol $\int$ is the integral sign, and the presence of $dx$ is a reminder of the inverse process of differentiation (anti-differentiation).

**Examples**

| **Function** $f(x)$ | **Primitive** $F(x)+C$ |
|---|---|
| $x^2$ | $\frac{1}{3}x^3+C$ |
| $\cos x$ | $\sin x+C$ |
| $\sin x$ | $-\cos x+C$ |
| $e^x$ | $e^x+C$ |
| $\frac{1}{x}$ | $\ln x+C$ |

Note that the function $G(x)=F(x)+C$, where $C$ is an arbitrary real number, is also a primitive of $f(x)$ because $G'(x)=F'(x)$.

This implies that a primitive of a function is not unique, however, any two different primitive functions of a given function differ only by a constant:

$F(x)-G(x)=C$.

**Rule Governing Primitive Functions**

If $F,G$ are primitive functions of $f,g$ on an interval $I$ then $F\pm G$ is a primitive of $f\pm g$.

**DO THIS**

Find the primitive functions of the following functions

1. $f(x)=3x^2+2x+1$; 2. $f(x)=\tan x$; 3. $f(x)=\sqrt{x+4}$

**Methods of Integration**

Look up the various methods listed below and ensure you master them. Discuss them with some colleagues and test yourselves by solving as many problems as you can in the exercises listed in the following section.

Integration by substitution
Integration by parts
Trigonometric integrals
Trigonometric substitution
Use of Partial Fractions
Numerical Integration

## 2.8 The Definite Integral of a function

**Definition:**
The definite integral of a **continuous** function $f(x)$, defined over a **bounded interval** $[a,b]$, is the **limit of a sequence of Riemann sums** $S_n = \sum_{i=1}^{n} \Delta x_i f(\xi_i)$, obtained by subdividing (partitioning) the interval $[a,b]$ into a number $n$ of subintervals $\Delta x_i = x_i - x_{i-1}$ in such a way that as $n \to \infty$ the largest subinterval in the sequence of partitions also tends to zero. The definite integral of $f(x)$ over $[a,b]$ is denoted by $\int_a^b f(x)dx$.

## Key Theorems and/or Principles

**Integrability of a function**
Every function $f$ that is continuous on a closed and bounded interval $[a,b]$ is integrable on $[a,b]$.

**Additivity of integrals**
If $f$ is integrable on $[a,b]$ and $a < c < b$ then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

**Fundamental Theorem of Calculus**
If $f$ is continuous on $[a,b]$ and $F(x) = \int_a^x f(t)dt$ for each $x$ in $[a,b]$, then $F(x)$ is continuous on $[a,b]$ and differentiable on $(a,b)$ and

$\frac{dF}{dx} = f(x)$. In other words, $F(x)$ is an anti-derivative of $f(x)$.

Proof of this theorem is relatively straightforward. It is based on the Mean Value Theorem for integrals stated above and on the definition of a continuous function $f(x)$ on a bounded interval. We state this result in the form of a Theorem:

Having defined the concept of a definite integral we now need to learn how to evaluate a definite integral.

Fortunately the concept of a primitive function introduced earlier gives us the answer. We state and show this important result in the form of a theorem.

**Theorem:** If $f(x)$ is continuous on a closed and bounded interval $[a,b]$, and $g^{'}(x) = f(x)$, then $\int_a^b f(x)dx = g(b) - g(a)$.

### Evaluation of Definite Integrals

Definite integrals can be evaluated in two ways. One method is using the anti-derivative, if it is known. A second method is by applying numerical methods.

### Evaluation of Definite Integrals Using Anti-derivatives

If the anti-derivative of $f(x)$ is $F(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$.

For example, since the anti-derivative of $f(x) = x^2$ is $F(x) = \frac{x^3}{3}$, it follows that $\int_2^3 x^2 dx = F(3) - F(2) = \frac{19}{3}$.

**DO THIS**

Evaluate the definite integral $\int_0^{\frac{\pi}{2}} \cos(x)dx$.

### Evaluation of Definite Integrals Using Numerical Methods

Numerical methods for evaluating definite integrals will be discussed in module 8 of this course.

## 2.9 Summary

We have covered quite a lot of ground so far in laying the ground for the mathematical problems that we shall develop mathematical methods of solving them numerically. The concepts of limit, continuity and differentiability are essential properties that we shall assume in all our efforts to carry out numerical integration. The learner is strongly encouraged to master these basic concepts well so that the numerical methods that will be developed have a sound mathematical basis for their applicability, validity and usefulness.