OIT 102 - Supplementary Notes

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Contents

1 Functions

A function is a mathematical object that is rather abstract, but which is never the less used in a very large number of mathematical applications. Before we can define a function, however, we need to be familiar with a variable.

1.1 Variables

A variable is in a sense a place-holder. The variables that we will meet here, will all hold the place of a real number. That it is a *variable* means, in the sense of the word, that it is not set and can be changed (varied) either by us or by some mathematical operation that is performed on it.

If there is no defining property of some variable except that it holds the place of a real number, it is called an *independent variable*. The independent variable is usually denoted by x. It is called independent because it doesn't depend on anything (other than the human being who may choose to assign different values to it).

If a variable is defined by some operation performed on another (independent) variable, then its value depends on the operation performed and on the value of the other variable, and we call it a *dependent variable*. The dependent variable is usually denoted by y.

Note that what is important here is to think of the variables, both the independent and dependent ones, as something that denotes (holds the place of) a real number. We choose to use variables and call them x and y because there are infinitely many real numbers, so if we had to talk about them one by one, we would never finish. We want to be able to perform operations on or discuss properties of any subsets of the real numbers, so we conveniently use a symbol that means "any real number (with a given property)".

1.2 Definition of a Function

A function is a mathematical object that takes an independent variable, performs some predefined operation on it, and returns the value of the dependent variable. An example could be:

$$\underbrace{y}_{\text{dependent variable}} = \underbrace{f(x)}_{\text{function name and independent variable}} = \underbrace{7 \cdot x^2 + 5 \cdot x - 2}_{\text{function body}}$$

As we can see, the *function body* is a recipe telling us how to get the value of the dependent variable for each choice of the independent variable. Note that as long as we haven't made a choice of independent variable, the function $f(x)$ is just a mathematical recipe.

Once we have chosen to assign a value to the independent variable, we automaticall get a value of the dependent variable that goes with it. We say that in the point x, the function evaluates to y, or that the function's value in x is y. Note that it is customary to have some notational confusion here as $f(x)$ can be meant to denote the mathematical recipe as well as the function's value in x. Usually we can tell from the context which is which.

1.3 Uniqueness of the Function's Value

For a mathematical recipe to be a function, we demand that it **uniquely defines** the function's value in any point. We cannot have a function such as:

$$f(x) = \begin{cases} x+3 & \text{on Mondays} \\ 5x-2 & \text{on Tuesdays} \\ -x^2 & \text{on Wednesdays} \\ 0 & \text{on Thursdays} \\ \frac{1}{2x} & \text{on Fridays} \end{cases}$$

Or such as the one given in the table below:

f(x)	1	1	3	4	5
x	3	4	5	3	5

Imagine using the first one in the construction of a house. It would be a very strange house indeed, changing its shape each day of the week and being undefined on weekends. I wouldn't want to live there!

The problem with the second is that it assigns 2 different values to the number 1. This violates the principle that a function must uniquely define the value that each choice of independent variable evaluates to.

2 Domain and Range

The domain of a function is the largest set of numbers that can be evaluated using the recipe given in the function body of that particular function. With a lot of functions, the domain is simply all of the the real numbers, $\mathbb{R}$. A point (a real number) is excluded from the domain of a function in the case that it will cause an illegal (mathematicall speaking) action if you try to evaluate the function in that point.

The most common illegal action is division by 0. We may under no circumstances divide by 0. To give an example of this, consider the function:

$f(x) = \frac{x}{x-1}$

If $x = 1$, the denominator becomes $1 - 1 = 0$, so the domain of this function cannot include the number 1. All other reals are fine, so in this case the domain of the function, often denoted $\mathcal{D}(f(x))$, becomes all the real numbers except 1, $\mathcal{D}(f(x)) = \mathbb{R} \setminus \{1\}$.

Note that we may for other purposes chose to define a smaller domain. In that case we must supply a specification of the domain over which we want to evaluate the function.

The range if a function is the set of numbers that the function can evaluate to over its domain. If our function is

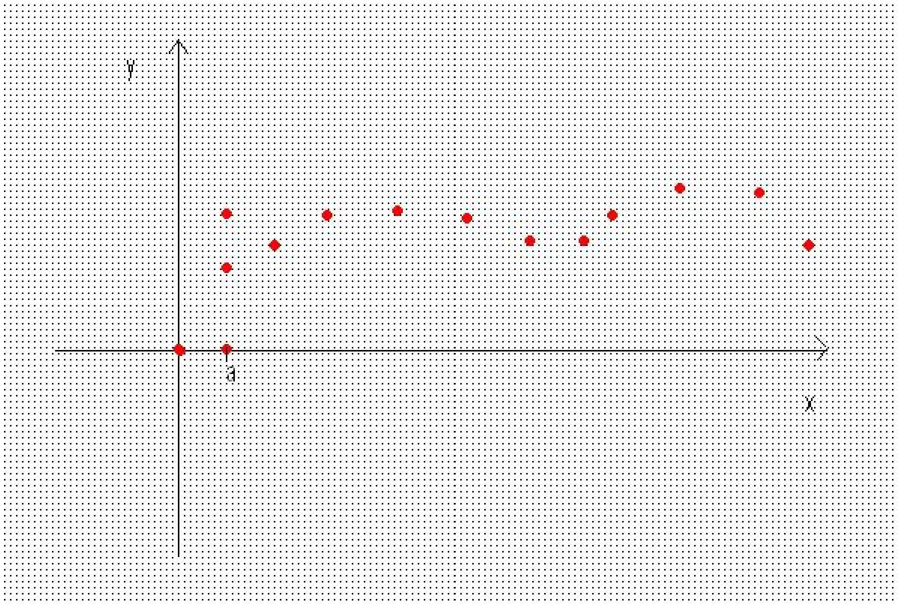
$f(x) = x + 2$

and we have specified the domain as $\mathcal{D}(f(x)) = \{1, 2, 3, 4, 5\}$, then the range of the function is the set $\mathcal{R}(f(x)) = \{3, 4, 5, 6, 7\}$. If we define the domain to be $\mathcal{D}(f(x)) = [1, 5]$ (the interval from 1 to 5), then we get $\mathcal{R}(f(x)) = [3, 7]$ (the interval from 3 to 7).

2.1 Graphs

The graph of a function is the graphical representation of that function, drawn in a coordinatesystem such that the points are the independent variable and its functional value, $(x, f(x))$. The graph will always be incomplete except in the cases where we have restricted our domain to a small part of the real line. When the domain hasn't been specified (and thus must be assumed to the all of the real line), we must remember that we look only at a part of the graphical represetation of that function.

We can determine if something is a function by looking at the graph. Given a situation in which any value of the x-axis has more than one second coordinate, we know that this isn't a function:

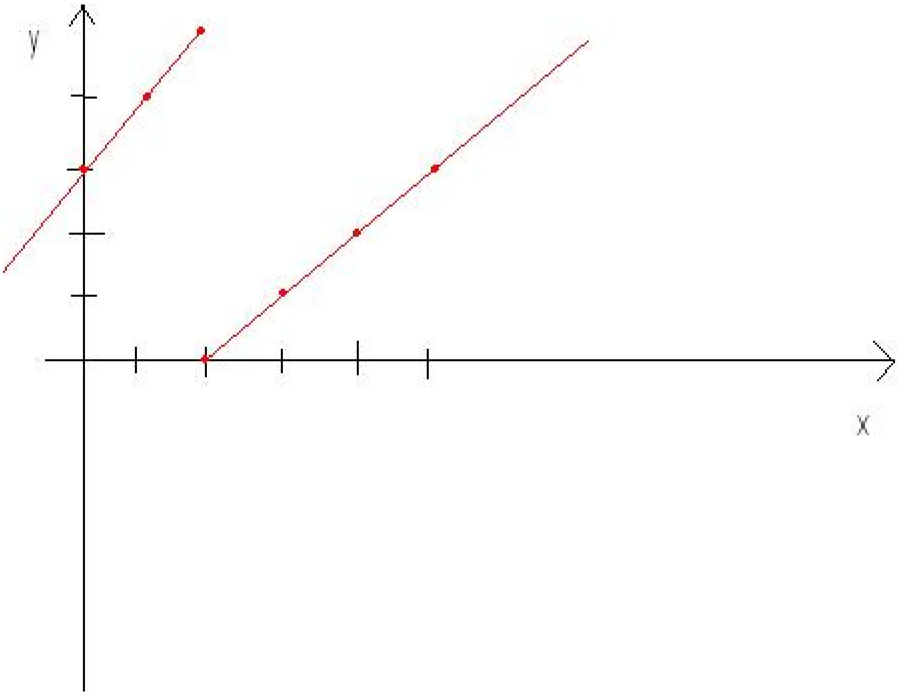


The above points cannot denote a function because there are 3 different second coordinates at the point a.

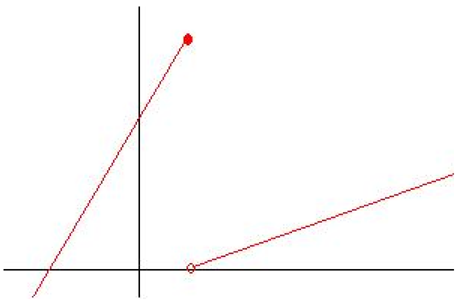
If we want to illustrate that the function makes a jump, we do as follows. Consider the function:

$$f(x) = \begin{cases} x+3 & x \leq 2 \\ x-2 & x > 2 \end{cases}$$

The graph would look like this:



And we use a closed dot to denote that the point $(2, 5)$ is included in the graph, whereas the point $(2, 0)$, denoted by an open dot, is not included in the graph:



If both of them had been closed, then the picture could not be of a graph of a function, and if both had been open, we would have a point in which the function isn't defined.

2.2 Composite functions

A function can easily be evaluated in a point that is the functional value of another function. If $f(x) = x + 2$ and $g(x) = 3x$, then we get the composite function $f(g(x))$ by inserting $g(x)$ on x's place: $f(g(x)) = (3x) + 2$. Note that $g(f(x)) = 3(x+2) = 3x+6$, so the order in which we call the functions does matter. We do sometimes write $f(g(x))$ as $f \circ g(x)$ or $(f \circ g)(x)$.

2.3 Questions

Problem 1: Which of the following tables can be functions:

f(x)	1	1	3	4	5
x	3	4	5	3	5

f(x)	1	3	4	5
x	3	5	3	5

f(x)	1	2	3	4	5
x	3	4	5	3	5

Problem 2: Define the domain of the following functions:

1) $f_1(x) = x - 3$

2) $f_2(x) = x^2 - 2x + 1$

3) $f_3(x) = \frac{1}{x}$

Problem 3: Given the domain $[0.5, 2]$, define the range of the following functions:

1) $f_1(x) = x - 3$

2) $f_2(x) = x^2 - 2x + 1$

3) $f_3(x) = \frac{1}{x}$

3 Straight Lines

A special kind of function is that describing a straight line. It has the form $f(x) = ax+b$, where a, typically a fraction, denotes the slope of the line, and

b denotes the point where the line crosses the y-axis. Here it is important to note the difference between the kind of variables x, a and b. The independent variable x may take any value of the real line, whereas the numbers a and b, though they may vary from straight line to straight line, do not change once we have assigned a value to them.

If a function is given by $f(x) = \frac{1}{3}x + 1$, we know that it crosses the y-axis at the point $(0, 1)$ (try inserting 0 into x's place and evaluate), and we know that each time we take 1 step up the y-axis, we must take 3 steps along the x-axis.

Given 2 points (x_1, y_1) and (x_2, y_2) in a coordinate system, we can always find the formula for the straight line connecting them as follows:

$a = \frac{\text{difference in y-coordinates}}{\text{difference in x-coordinates}} = \frac{y_2 - y_1}{x_2 - x_1}$.

As the general formula is $f(x) = ax + b$, we can find b by inserting one of the known pair of coordinates as follows:

$y_1 = a \cdot x_1 + b$ to get $y_1 - a \cdot x_1 = b$, or $y_2 - a \cdot x_2 = b$ if we prefer those numbers to the others.

3.1 Questions

Problem 1: Find the formulas for the straight lines that goes through the following sets of points:

1) $(1, 2)$ and $(3, 4)$

2) $(2, 3)$ and $(4, 1)$

3) $(-2, -4)$ and $(2, 4)$

Problem 2: Given the slope $a = \frac{1}{2}$, find the formulas for the straight lines that goes through the following points:

1) $(1, 2)$

2) $(4, 1)$

3) $(-2, -4)$

4 Logarithms

The (base 10-) logarithm of a number is the power of 10 (as in the exponent) that will give us that number. A few examples:

$10^0 = 1$ and $\log(1) = 0$. $10^1 = 10$, and $\log(10) = 1$. $10^2 = 100$, and $\log(100) = 2$. $10^3 = 1000$, and $\log(1000) = 3$.

Of course we can take the logarithm of a number that's not a power of 10, but we'd need a logarithmic table or a calculator to find it.

The opposite operation of the base 10-logarithm of x is to lift 10 to the power of x, e.g. $\log(10,000) = 4$ and $10^4 = 10,000$.

The natural logarithm is a logarithm that has the number e as it's base. The number e is approximately 2.71828. There's a reason for this seemingly random base, but we won't get into that now. The natural logarithm to x is written $\ln(x)$.

Logarithms are nice because they make complicated mathematical operations quite straight forward. We have that, irrigardless of the base of the logarithm:

1) $\ln(x \cdot y) = \ln(x) + \ln(y)$

2) $\ln(\frac{x}{y}) = \ln(x) - \ln(y)$

3) $\ln(x^y) = y \cdot \ln(x)$

4) $\ln(\frac{1}{x}) = -\log(x)$

5a) For the natural logarithm: $\ln(e^x) = x$ and $e^{\ln(x)} = x$

5b) For the base 10-logarithm: $\log(10^x) = x$ and $10^{\log(x)} = x$

4.1 Questions

Problem 1: Using the rules for calculating with logarithms given above, calculate the following numbers. Expressions such as $\log(n)$ when n is not a power of 10 are acceptable:

1) $\log(3,000) = \log(1000 \cdot 3)$

2) $\log(27) = \log(3^3)$

3) $\log(54) = \log(2 \cdot 3^3)$

4) $\log(20) = \log(\frac{2\cdot 100}{10})$

5) $\log(e^{\ln(5)})$

5 Limits

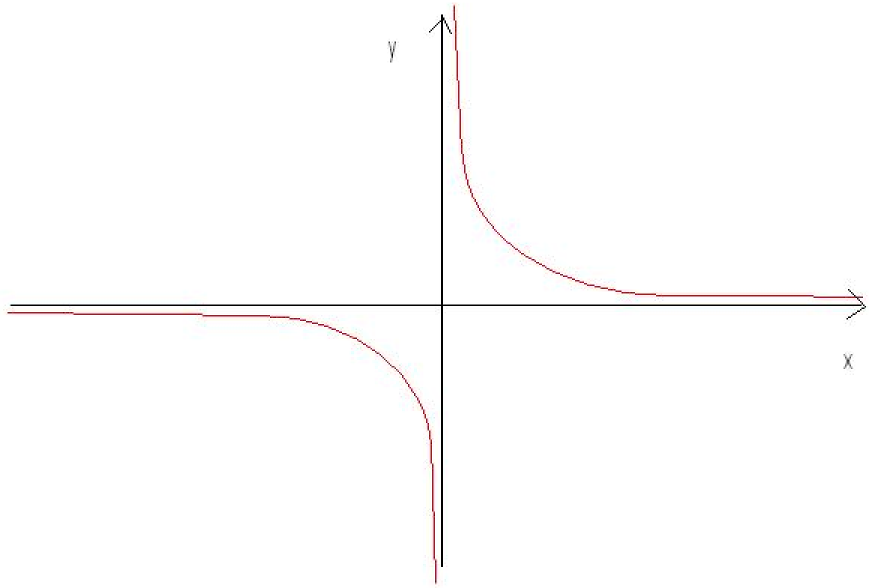
The limit of an expression including some variable means the value, if it exists, that the expression evaluates to as the variable approaches some number. If the expression is called $E(x)$ and we know that $E(x)$ tends to (approaches) L as x tends to a, we write it $lim_{x\to a}(E(x)) = L$.

Some examples are:

$lim_{x\to 2}(x^2) = 4$

$lim_{x\to 3}(x + 2) = 5$

We may exprience situations when the limit doesn't exist. For instance, consider function $f(x) = \frac{1}{x}$. Its graph looks as follows:



Which value should we assign to $lim_{x\to 0}(f(x))$? Obviously, as x tends to 0 from the left, $f(x)$ tends to $-\infty$ (we may write this $lim_{x\to a^-}(f(x)) = -\infty$), but as x tends to 0 from the right, $f(x)$ tends to ∞ (we may write this $lim_{x\to a^+}(f(x)) = \infty$). And as $-\infty \neq \infty$, we cannot assign a value to the limit.

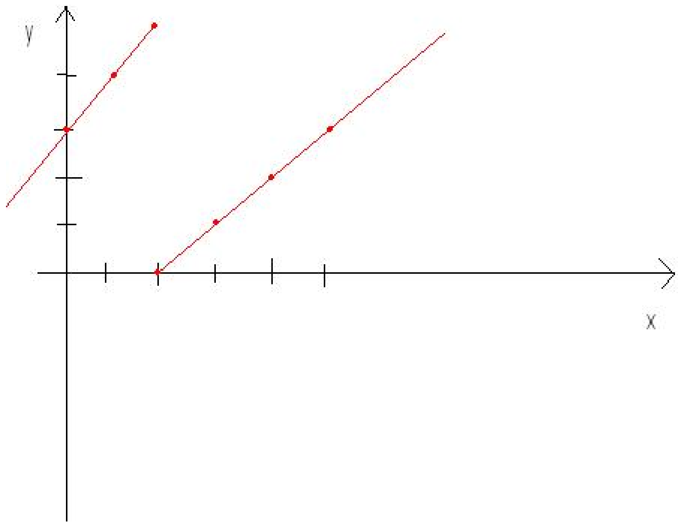
In other cases of division by 0, we may be more lucky and find that the expression actually has a limit. Consider the expression $\frac{x}{x-1}$. What happens when x tends to 1? In this case, $lim_{x\to 1}(\frac{x}{x-1})$ exists and equals 1.

Note that though we can't actually calculate $\frac{x}{x-1}$ for $x = 1$ as this would entail division by 0, we can still find the limit as x tends to 1.

6 Continuity

Returning to functions, we say that a function $f(x)$ is continuous in the point a if $lim_{x\to a}(f(x)) = L$ and $f(a) = L$ no matter whether we approach a from the left or from the right, or in other words, $f(x)$ is continuous in a if $lim_{x\to a^-}(f(x)) = lim_{x\to a^+}(f(x)) = L$.

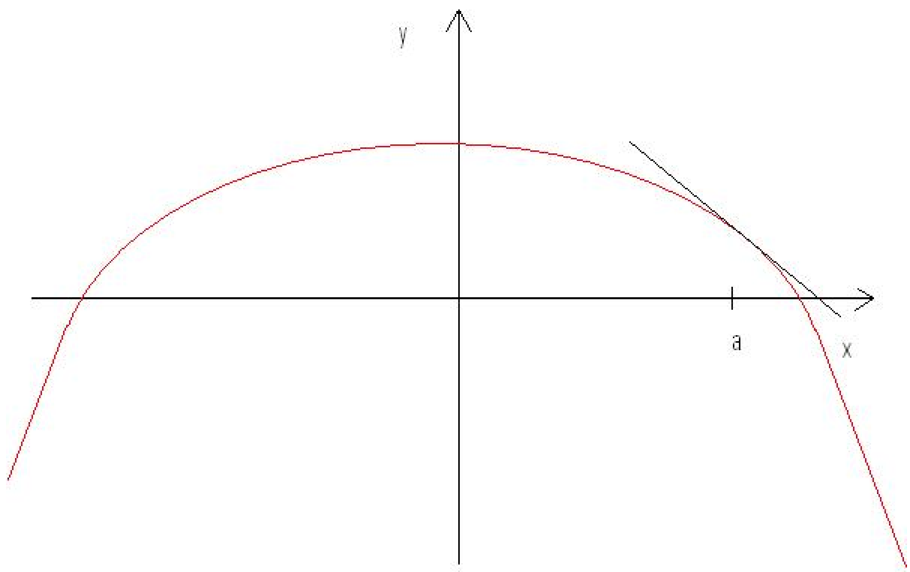
In the case of the real numbers, a continuous function can easily be identified by its graph. If you can draw the graph without lifting your pencil from the paper, then the function is continuous. Thus, our function from earlier:



is discontinuous in the point $x = 2$, but continuous over the intervals $]-\infty, 2]$ and $]2, \infty[$.

7 Derivatives

We can only find the derivative of a function $f(x)$ in the points $x = a$ where $f(x)$ is continuous. In this case, the derivative of $f(x)$ in the point $x = a$ is the slope of the tangent line of the graph of the function in the point $(a, f(a))$.



This is done by calculating the limit $\lim_{h \to 0}(\frac{f(x+h)-f(x)}{h})$ (refer to the formula for finding the slope of a straight line between 2 points). We write the derivative of $f(x)$ as $\frac{d}{dx}(f(x))$ or $f'(x)$.

There are various rules for calculating derivatives. The following table is not exhaustive and the reader is strongly encouraged to look up more rules for calculation on the internet (for example here: `http://en.wikipedia.org/wiki/List_of_differentiation_identities`) or in other textbooks. The letter k denotes some constant:

$f(x)$	$f'(x)$
k	0
x	1
$k \cdot x$	k
x^n	$n \cdot x^{n-1}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\frac{1}{x}$	$\ln(\|x\|)$
e^x	e^x
e^{kx}	ke^{kx}
$\cos(x)$	$-\sin(x)$
$\sin(x)$	$\cos(x)$

The following rules tells us how to find derivatives of composite functions. Let $g(x)$ denote some other continuous function:

$kf(x)$	$kf'(x)$	(multiplication by a constant)
$f(x)+g(x)$	$f'(x)+g'(x)$	(sum rule)
$f(x)g(x)$	$f'(x)g(x)+f(x)g'(x)$	(product rule)
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$	(fraction rule)
$f(g(x))'$	$f'(g(x))g'(x)$	(chain rule)

7.1 Questions

Problem 1: Find the derivatives of the following functions:

1) $f_1(x) = 2x + 3$

2) $f_2(x) = 5x + 2$

3) $f_3(x) = 4x^2 + 4x + 4$

4) $f_4(x) = e^x + 2x$

5) $f_5(x) = e^{2x} + 2x$

6) $f_6(x) = 2xe^x$

7) $f_7(x) = \frac{e^x}{2x}$

8) $f_8(x) = \sin(2x)$

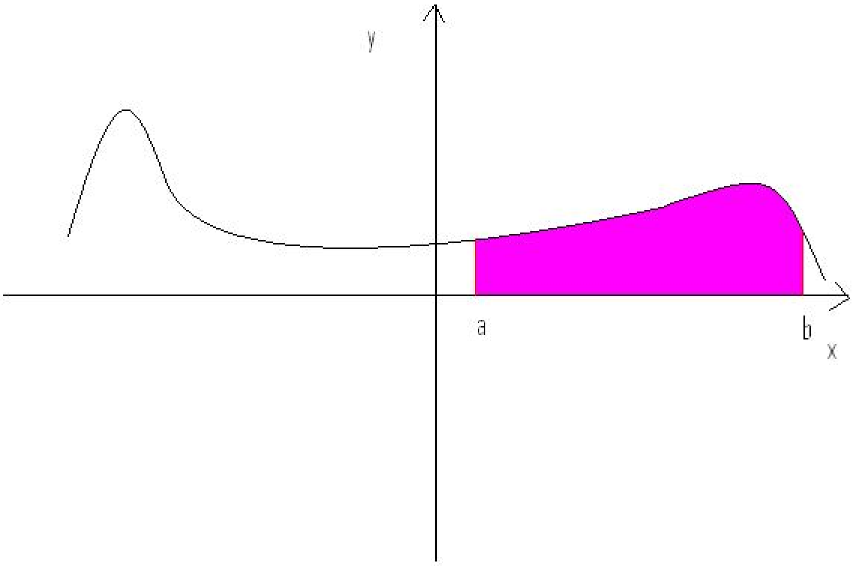
9) $f_9(x) = \cos(2x + 3)$

10) $f_{10}(x) = \frac{\sin(x)}{e^x}$

8 Integration

Integration is the opposite of derivation. If for some continuous function $F(x)$ we have that $F'(x) = f(x)$, then we also have that $\int f(x)dx = F(x)$. This is also called the indefinite integral of $f(x)$. It is indefinite because the result is another function.

If we have a continuous function, what we do graphically when we integrate, is to calculate the area under the graph of the function:



We say that this is the definite integral of $f(x)$ from a to b, and we write it $\int_a^b f(x)dx$. It is definite because the result is a number. We have that $\int_a^b f(x)dx = F(b) - F(a)$.

The reader is strongly encouraged to look up the rules of integration on the internet (for example here: `http://en.wikipedia.org/wiki/Lists_of_integrals`) or in another textbook. The following table contains a few of them:

$f(x)$	$F(x)$
k	kx
x	$\frac{1}{2}x^2$
$k \cdot x$	$\frac{k}{2}x^2$
x^n	$\frac{1}{n+1}x^{n+1}$
e^x	e^x

And here are a few of the rules on how to calculate integrals:

1) $\int kf(x)dx = k\int f(x)dx$

2) $\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx$

3) $\int f(x)g(x)dx = F(x)g(x) - \int F(x)g'(x)dx$

4) $\int f(g(x))g'(x)dx = \int f(u)du$, $u = g(x)$

8.1 Questions

Problem 1: Calculate the following definite integrals:

1) $f_1(x) = 2x + 3$

2) $f_2(x) = 5x + 2$

3) $f_3(x) = 4x^2 + 4x + 4$

4) $f_4(x) = e^x + 2x$

5) $f_5(x) = 2e^x + 2x$

Problem 2: Calculate the following indefinite integrals:

1) $f_1(x) = 2x + 3$

2) $f_2(x) = 5x + 2$

3) $f_3(x) = 4x^2 + 4x + 4$

4) $f_4(x) = e^x + 2x$

5) $f_5(x) = 2e^x + 2x$