Module 2: More basic set operations

September 17, 2009

Contents

1 Introduction:

Sets and set operations appear in a large number of mathematical concepts and methods as well as forming a basis for many operations that we do not normally connect with sets. We were introduced to the union and the intersection of 2 sets in the last module, and here we will be introduced to more basic set operations, which are just as important.

This module corresponds to sections 1.6 and 1.7 in the chapter on sets.

2 Objectives:

At the end of this module the learner will be able to:

- Form the complement of a set

- Given any 2 sets, identify the intersection, union, difference and symmetric difference of them

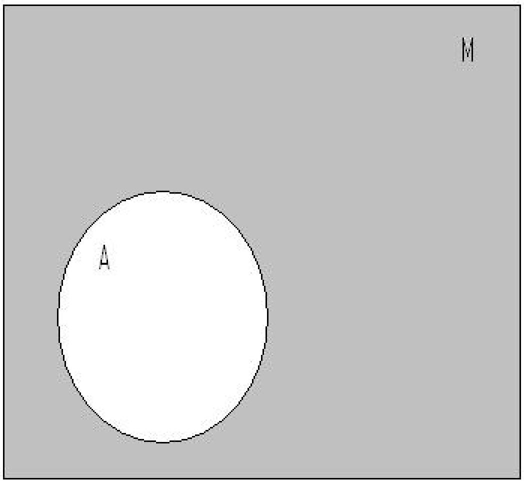
- Use Venn diagrams to explain more set theoretic concepts

- Use the laws of set algebra

- Use the concept of duality to explain simple set theoretic identities

3 Complement of a Set:

Using Venn diagrams to illustrate the complement of a set, we get that for a set A, the complement of A, written A^C, is the set of all the elements of the universal set U that are not elements of A:

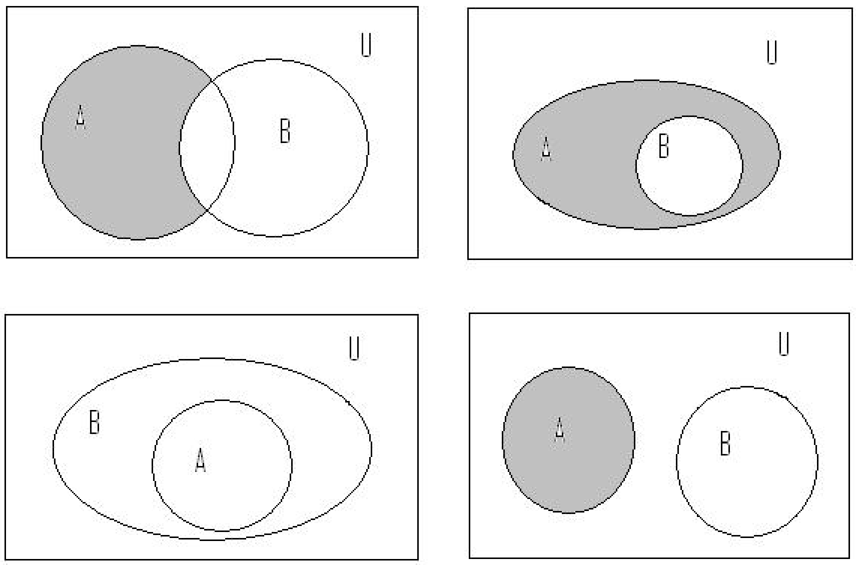


The shaded area is the complement of A, A^C.

4 Set Difference:

The set difference between 2 sets A and B is written $A \setminus B$ and denotes the elements of U that are elements of A, but are not elements of B.

Using Venn diagrams to illustrate the concept:

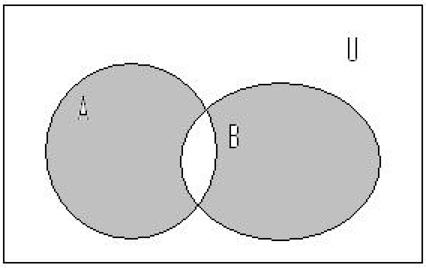


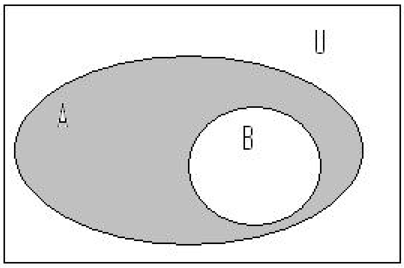
All 4 Venn diagrams illustrate set differences. Note that when A is completely conrtained in B, the set difference $A \setminus B$ is the empty set, $\emptyset$, and when A and B have no elements in common, the set difference $A \setminus B$ is simply A. In terms of complements, we see that $A^C = U \setminus A$.

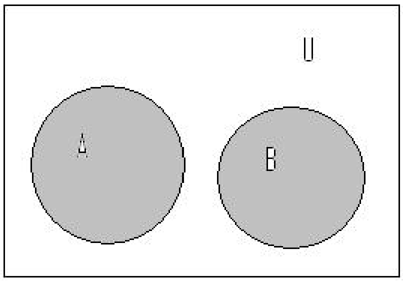
Note, too, that $A \setminus B$ is not the same as $B \setminus A$.

5 Symmetric Difference:

Recall from last module that the union of 2 sets A and B consists of all the elements of the universal set U that are elements of A or B or both. The symmetric difference between 2 sets A and B consists of all the elements of $A \setminus B$ and all the elements of $B \setminus A$, or in symbols: $(A \setminus B) \cup (B \setminus A)$. We write the symmetric difference between A and B as $A \triangle B$ or $A \oplus B$.







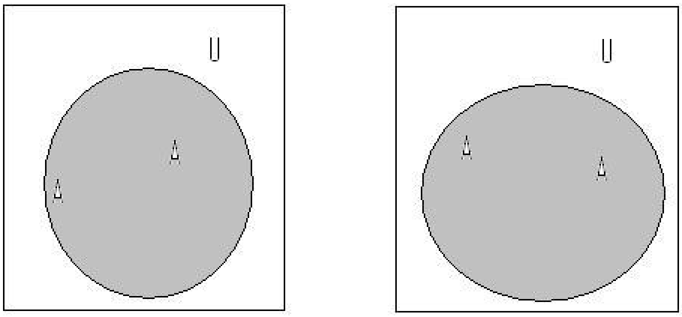
Note that if all the elements of B are contained in A, the symmetric difference is the same as $A \setminus B$, and if A and B have no elements in common, the symmetric difference is the same as $A \cup B$.

Note, too, that $A \triangle B = B \triangle A$, which is why it is called a symmetric difference.

6 The Algebra of Sets:

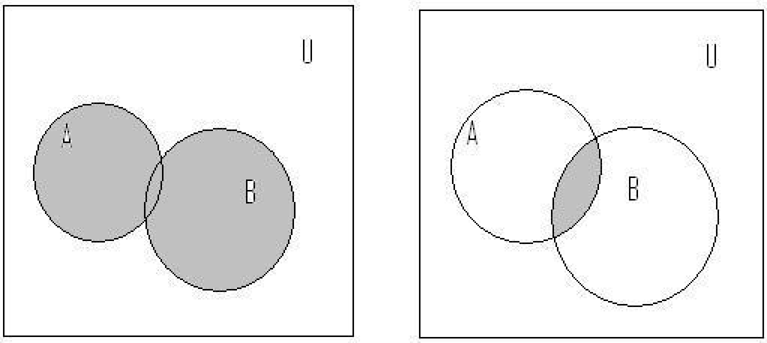
Recall from last module that the intersection of 2 sets A and B are the elements of the universal set U which are elements of both A and B.

As noted with a lot of the set operations, the order in which you apply them is very important. The operations for which the order does not matter are the following:



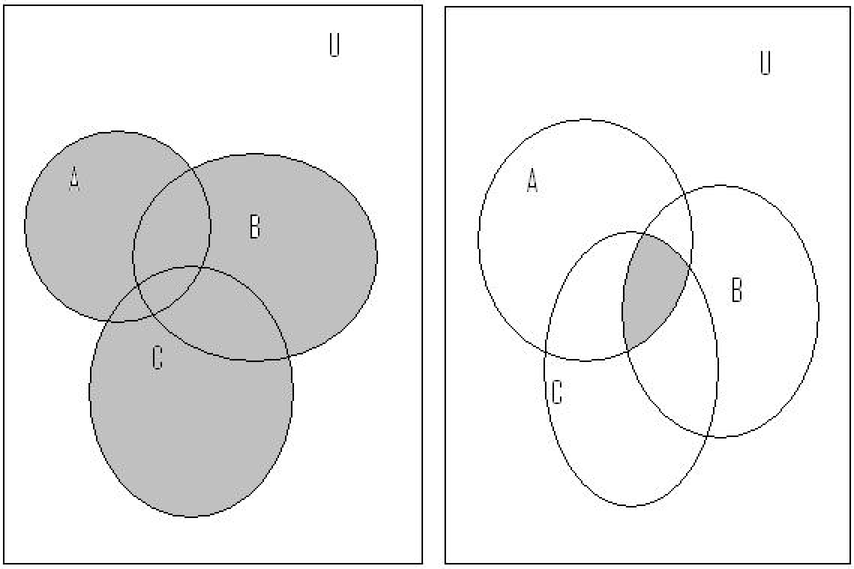
$A \cup A = A$ and $A \cap A = A$
Idempotent Laws

That an operation is commutative means that you can swap the order of the operands:



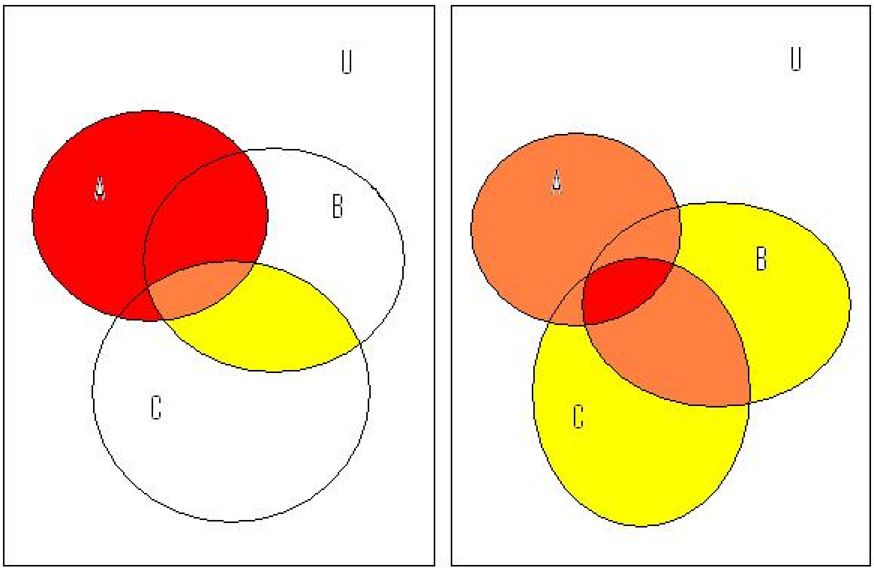
$A \cup B = B \cup A$ and $A \cap B = B \cap A$
Commutative Laws

That an operation is associative means that when you apply it to more than 2 operands, the order in which you apply it doesn't matter:

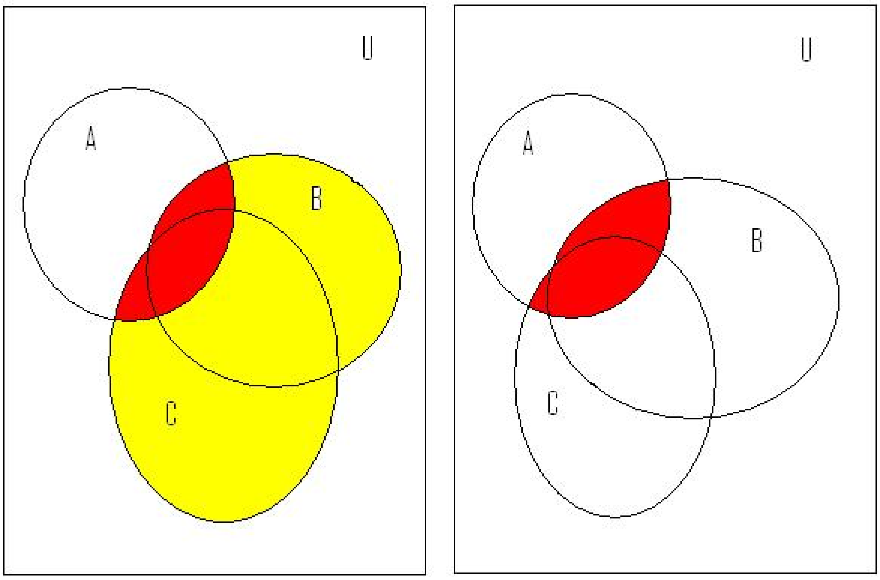


$A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$
Associative Laws

The Distributive Laws tell us how the operations are distributed with respect to each other:



$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Distributive Law



$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Distributive Law

We also have the following identities, which are difficult to draw as Venn diagrams, but which are very obvious once you think about them:

$A \cup \emptyset = A$ (Identity Law)

$A \cap U = A$ (Identity Law)

$A \cup U = U$ (Identity Law)

$A \cap \emptyset = \emptyset$ (Identity Law)

$(A^C)^C = A$ (Involution Law)

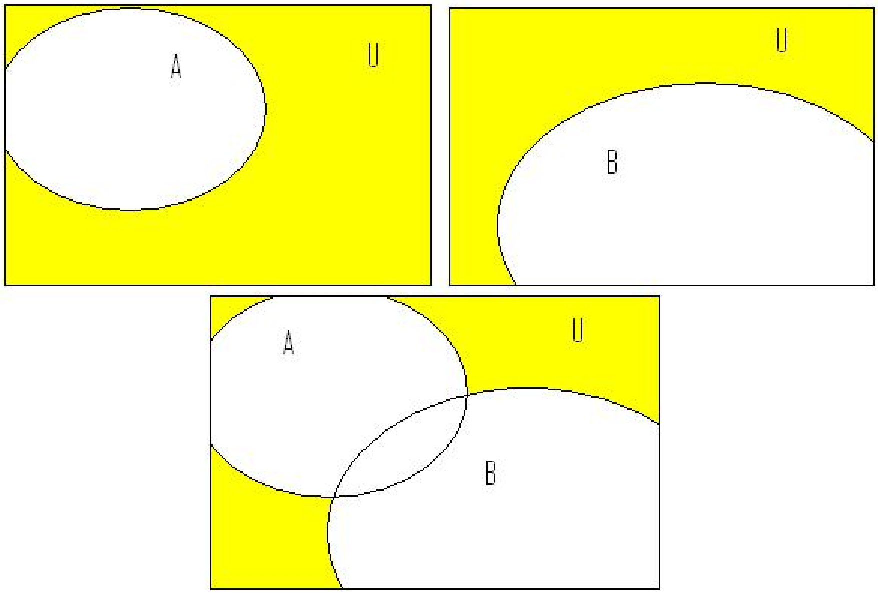
$A \cup A^C = U$ (Complement Law)

$A \cap A^C = \emptyset$ (Complement Law)

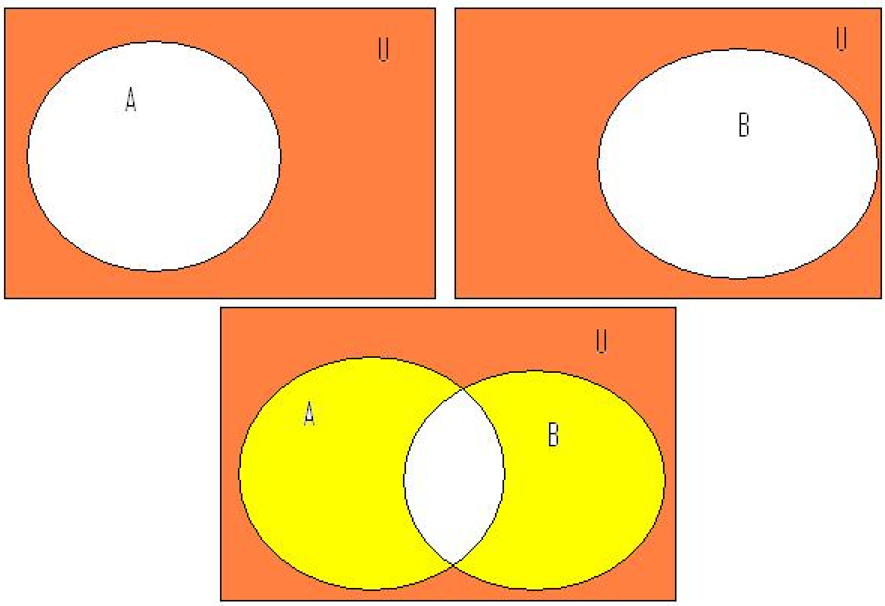
$U^C = \emptyset$ (Complement Law)

$\emptyset^C = U$

Finally, we have De Morgans Laws, which are very important in (formal) logic. Here, we meet them in their set theoretic interpretation:



$(A \cup B)^C = A^C \cap B^C$
De Morgan's Law



$(A \cup B)^C = A^C \cap B^C$
De Morgan's Law

7 Duality

Duality is a core concept of set theory. We get the dual of a statement by replacing the operators pairwise as follows:

$\cup$ is replaced by $\cap$

$\cap$ is replaced by $\cup$

U is replaced by $\emptyset$

$\emptyset$ is replaced by U

The good thing about duality is that if you have an identity, the dual of that identity is automatically an identity, too.

To get the dual in the exemple of the book, we do as follows, evaluating the parentheses first:

START: $(U \cap A) \cup (B \cap A)$

$U \to \emptyset, \cap \to \cup, A = A$ and $B = B, \cap \to \cup, A = A$, so we get:

$(U \cap A) \to (\emptyset \cup A)$ and $(B \cap A) \to (B \cup A)$

Having dealt with the parentheses, we get $\cup \to \cap$, so the dual becomes:

$(\emptyset \cup A) \cap (B \cup A)$.

Using the laws of set algebra from the last section, we see that:

$(\emptyset \cup A) = A$ and $A \cap (B \cup A) = A$, or with the dual expression:

$(U \cap A) = A$, and $A \cup (B \cap A) = A$. For the last identity we have used that all elements of $A \cap B$ are elements of A, so that we do not add anything that wasn't an element of A to begin with.