Module 3: Finite Sets, Counting Principles

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Contents

1 Introduction:

Sets and set operations appear in a large number of mathematical concepts and methods as well as forming a basis for many operations that we do not normally connect with sets. Counting the elements of any number of finite sets is easy as long as the sets are disjoint (have no elements in common), but in case there are elements which belong to more than one set, it becomes more complicated. Luckily, there are formulas to assist us counting the elements even if they belong to more than one set.

This module corresponds to section 1.8 in the chapter on sets.

2 Objectives:

At the end of this module the learner will be able to:

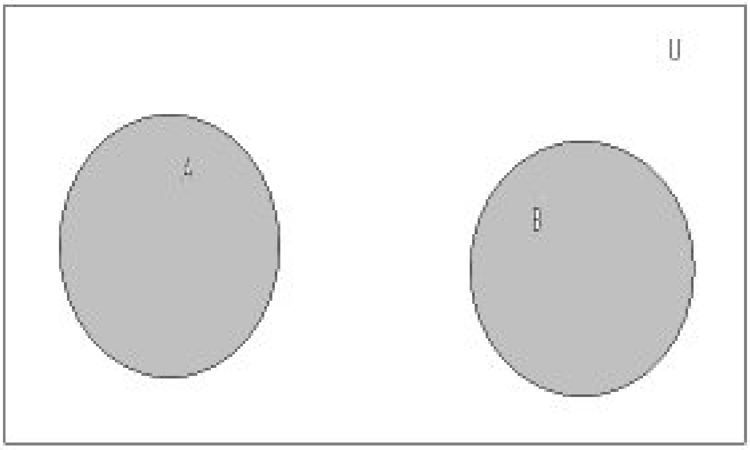
- Count the elements of a collection of any number of finite sets

3 Finite Sets:

A set is called finite if it has a finite amount of elements. A finite amount means that if you count them, you will finish before the end of time. If a set has a finite amount of elements, counting these elements make sense, but if it hasn't, you will never finish your task if you set out to count them.

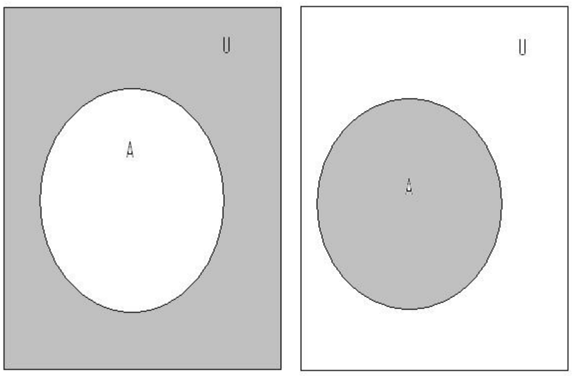
4 Counting the Elements of Disjoint Sets:

That 2 sets are disjoint means that they have no elements in common:



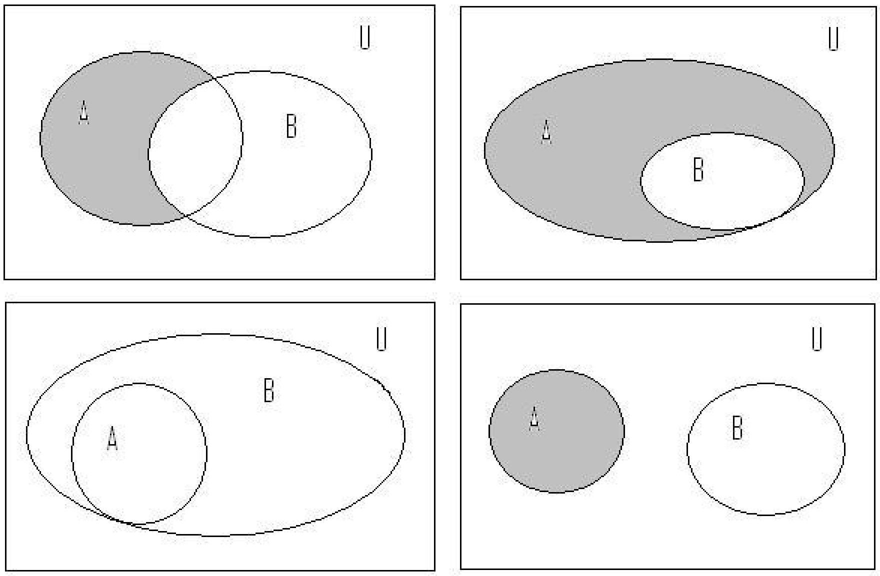
In this case, we simply count the elements of one set and add that number to the number of elements in the other set. Denoting the number of elements of a set A by $n(A)$, we get that as long as A and B are disjoint sets, then $n(A \cup B) = n(A) + n(B)$.

Similarly, if we want to count the elements of the universal set U, assuming that it is finite, we get that $n(U) = n(A) + n(A^C)$:



From this, we can easily deduct that $n(A^C) = n(U) - n(A)$ and $n(A) = n(U) - n(A^C)$.

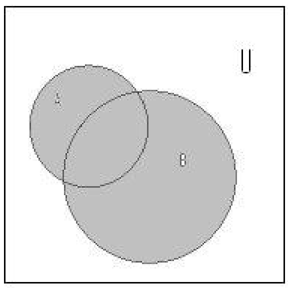
Further, we have that for 2 finite sets A and B, $n(A \setminus B) = n(B) - n(A \cap B)$:



Note that if A is completely contained in B, there are no elements in $A \setminus B$, and if A and B are disjoint, $n(A \setminus B) = n(A)$.

5 Counting Elements of Sets that are not Disjoint:

If we want to count the elements of sets that are not disjoint, we have to be a little more clever:

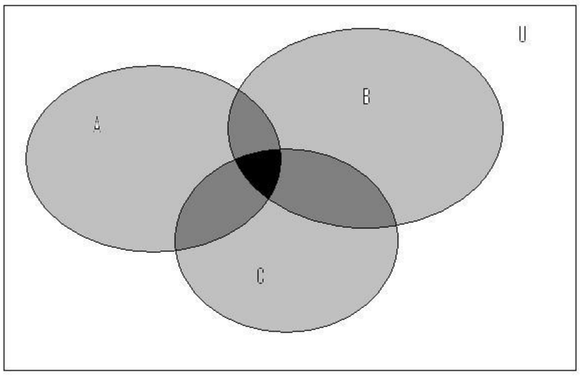


If we just use the same principle as with disjoint sets, we will have counted the elements that are in both A and B twice, so that approach doesn't work. What we do is to count them twice and them subtract them once:

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Note that this is the same formula as with the disjoint sets, as the intersection of disjoint sets is empty by definition and thus contains no elements.

Similarly, if we have 3 sets with a non-empty intersection, we add the elements of each set, subtract the elements that we have added twice, but now we will have subtracted the elements that are in all 3 sets one time too many, so we add those:



$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$