Module 4: Sets and (formal) Logic

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Contents

1 Introduction: **2**

2 Objectives: **2**

3 Power Sets and Classes of Sets **2**

4 Using Venn Diagrams to Justify Arguments: **3**

5 Mathematical Induction **6**

1 Introduction:

Sets and set operations appear in a large number of mathematical concepts and methods as well as forming a basis for many operations that we do not normally connect with sets. In this module, we look at how sets and logic are connected.

This module corresponds to sections 1.9 and 1.10 in the chapter on sets.

2 Objectives:

At the end of this module the learner will be able to:

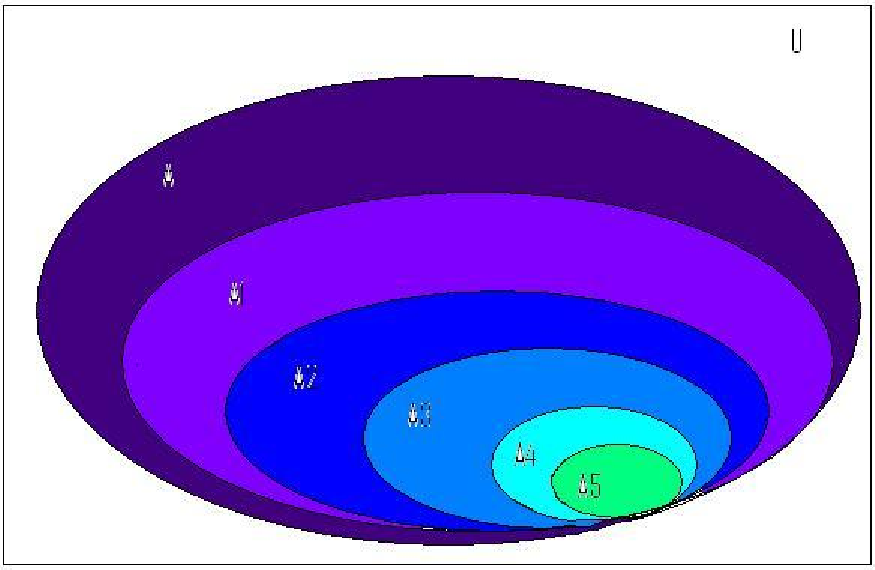
- Distinguish between power sets and classes of sets

- Use sets and Venn diagrams to justify arguments

- Use mathematical induction on the set of natural numbers

3 Power Sets and Classes of Sets

Recall from the first module that the statement $A \subseteq B$ (A is a subset of B) means that if you have an element of A, it is also an elements of B:



$$A5 \subseteq A4 \subseteq A3 \subseteq A2 \subseteq A1 \subseteq A$$

Note that the relation $A5 \subseteq A4$ automatically means that $A5$ is a subset of all the sets that $A4$ is a subset of: $A5 \subseteq A3$, $A5 \subseteq A2$, $A5 \subseteq A1$ and $A5 \subseteq A$. When a relation has the character, we say that it is transitive.

The power set of a set is the set of all subsets of that set. As a matter of convention, the empty set $\emptyset$, is always an element pf the powerset of a set. We write the powerset of a set A as $\mathcal{P}(A)$. In our case above (the Venn diagram), the power set of A has around 30 elements though we only have the set A and it's 5 subsets. This is because we must also consider all possible combinations of subsets of A, for instance $(A \setminus A1) \cup (A2 \setminus A3) \cup (A4 \setminus A5)$ will be in it.

The reason why we include the empty set and the whole set is not apparent here, but it will allow us to perform any set operation on A and always get a result that is an element of $\mathcal{P}(A)$.

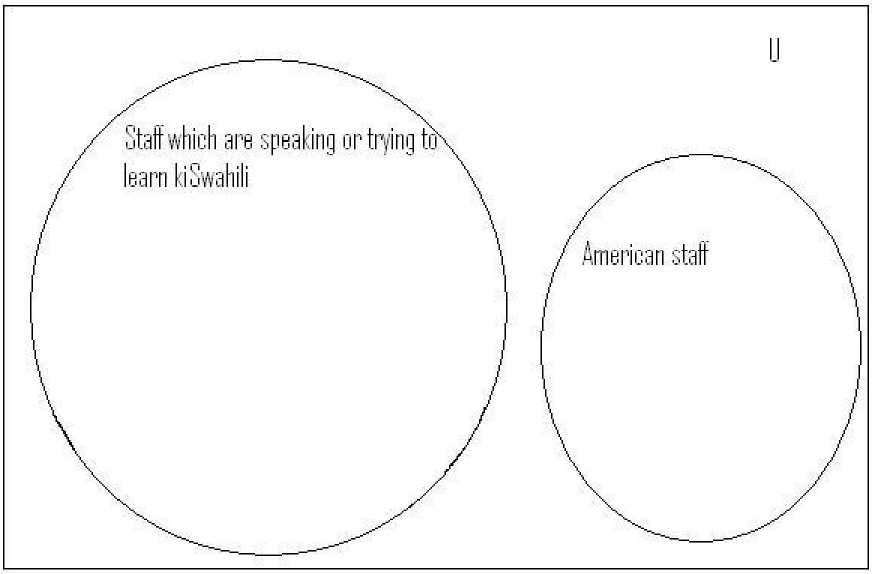
Note that is is perfectly possible for the elements of a set to be sets. However, we may speak of a class of sets or a collection of sets instead of speaking of a set of sets in order to avoid confusion.

4 Using Venn Diagrams to Justify Arguments:

The examples of using Venn diagrams to validate arguments shown in the book are very good and should be studied in detail. The task is to imagine some (undescribed) universal set that everything you argue about is contained in and determine the correct subsets which describes the varius subject matters of the argument.

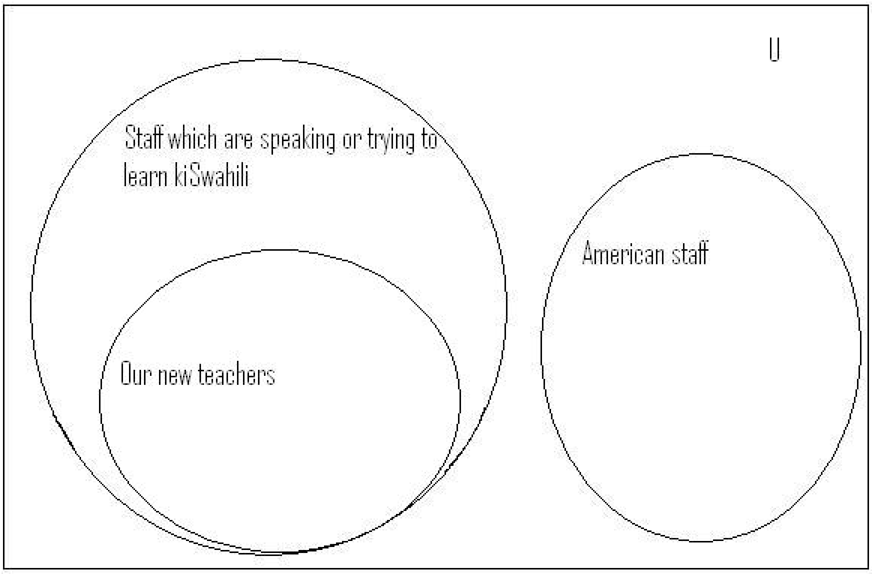
The undescribed universal set that everything we talk about will be part of

We might then add a statement such as "None of the American staff are trying to learn kiSwahili"

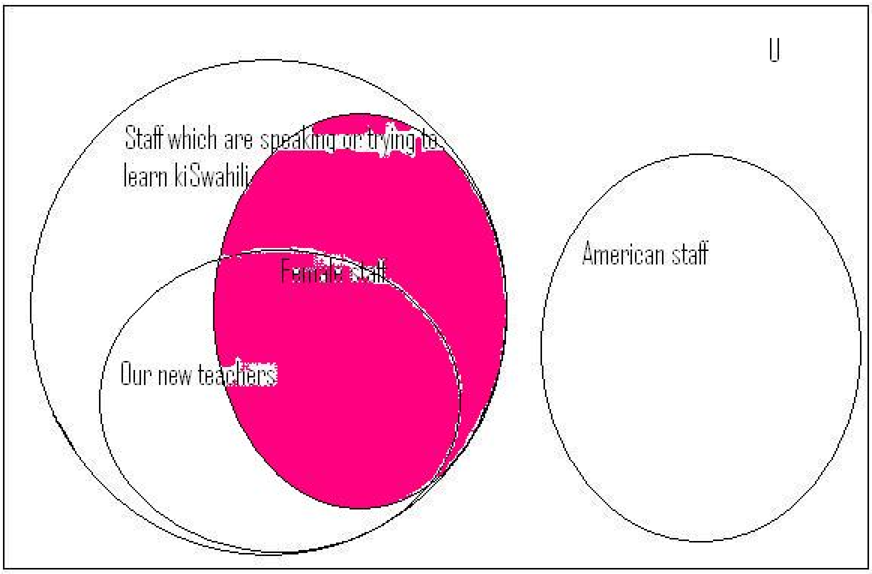


Note that we have to add the set of staff speaking or trying to learn kiSwahilli to show that the intersection between it and the set of American staff is empty.

Now, we can observe that "All our new teachers speak kiSwahili or are learning it"



This way, we can very clearly see that none of our new teachers are American. We can not, however, know if they're Tanzanian, Danish, Chinese or anything else. If we add another statement such as "All female staff speak or are learning kiSwahili",:



we can learn that none of the American staff are female (the set of female staff being pink), but we do not know if all our new teachers are female, or if we have teachers that are not new and who don't speak kiSwahili. We need more information to say anything about that.

5 Mathematical Induction

We did start out declaring that a set is completely determined by its elements. Mathematical induction is a principle of proving statements about natural numbers using this fact.

The natural numbers or the counting numbers is the set $\{1, 2, 3, 4, 5, \dots\}$, where we take the dots to mean that we continue in the same manner. The natural numbers is an infinite set, so we can't prove anything about them by proving it for one element at a time, we would never finish.

What we do is to observe certain things about the nature of the elements of this set:

1) We start with 1, and then we get each new element by adding 1.

2) We can immediately compare any 2 elements to each other and determine which is greatest and which is smallest. If a set has this characteristic, we call it completely ordered.

Note that any set with these 2 characteristics must be more or less identical to the natural numbers. As it is the elements of the set of natural numbers that give it these characteristics, we see how a set and its nature is determined by its elements.

Now, to prove a statement about the natural numbers, we use these characteristics as follows:

We start by checking that the statement is true for 1, this is called the START of the induction.

We then assume that statement is true some natural number n or $n - 1$. This is called the INDUCTION STEP.

Finally, we prove that if it holds for n, then it will also hold for $n + 1$

(alternatively, if it holds for $n-1$, then it also holds for n). This is the PROOF BY INDUCTION.

This way of proving a statement is valid because you can get any new natural number by adding 1 to the last one you had, and we have shown that if it holds for any natural number n, then it also holds for $n+1$. The reason why we need to be able to compare all numbers to each other is that we can't keep track of what is happening if we can get from the first natural number, 1, to any other natural number by adding 1 an appropriate amount of times.

To show another example than the one in the textbook, we claim that: "The sum of the first n natural numbers equals $\frac{n^2+n}{2}$, or in symbols: $\sum_{i=1}^{n} i = \frac{n^2+n}{2}$.

START: Check the claim for $n = 1$: $\frac{1^2+1}{2} = 1$. Now, this requires feeling comfortable with the sum of one element. If you don't like that, we can check for $n = 2$: $\frac{2^2+2}{2} = 3 = 1 + 2$, so it also holds for $n = 2$.

INDUCTION STEP: Assume that it holds for any set, but arbitrary, natural number n that $\sum_{i=1}^{n} i = \frac{n^2+n}{2}$.

PROOF BY INDUCTION:

$\sum_{i=1}^{n} +(n+1) = \frac{n^2+n}{2} + (n+1)$, so
$\sum_{i=1}^{n} +(n+1) = \frac{n^2+n}{2} + \frac{2(n+1)}{2}$, so
$\sum_{i=1}^{n} +(n+1) = \frac{n^2+n+2n+2}{2}$, so
$\sum_{i=1}^{n} +(n+1) = \frac{n^2+2n+1+n+1}{2}$, so
$\sum_{i=1}^{n} +(n+1) = \frac{(n+1)(n+1)+n+1}{2}$, so
$\sum_{i=1}^{n} +(n+1) = \frac{(n+1)^2+(n+1)}{2}$,

which is exactly what we wanted to prove.

The other way of proving something by using mathematical induction is really the same, except the induction step is slightly different. Instead of assuming the claim to hold for some set, but arbitrary n, you assume that it holds for all the elements of the set $\{1, 2, 3, \ldots, n\}$, and then you show that if that is true, then the claim also holds for $n+1$.