Module 6: Implications and Tautologies

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Contents

1 Introduction: 2

2 Objectives: 2

3 Implications 2

4 Bi-implications 3

5 Tautologies 3

6 Logical Implication 4

7 Logical Equivalence 5

1 Introduction:

Logic is the basis of how we mathematicians argue. In order to be sure that an argument or a proof holds (is correct), we must follow the rules of (formal) logic in every step of the justification of it. The mathematical logic, or (formal) logic, doesn't look very similar to what we call logic in our everyday lives, but as we get more comfortable with it, we'll soon recognise that we use the same logical structures to justify arguments in spoken English - or kiSwahili.

This module corresponds to sections 1.3 and 1.4 in the chapter on logic.

2 Objectives:

At the end of this module the learner will be able to:

- Distinguish between a logical implication and an implication

- Understand tautologies and work with them

- Use truth tables to sow logical equivalence

3 Implications

If we think about everyday language, we may argue that A leads to B, B leads to C, and C leads to D, or that A implies B, B implies C, and C implies D. In math, nearly all argumentation follows the same pattern. Given 2 propositions p and q, we write $p \rightarrow q$ to say that p implies q. We call p the hypothesis and q the conclusion.

The truth table for $p \rightarrow q$ is as follows:

p	q	$p \rightarrow q$
True	True	True
True	False	False
False	True	True
False	False	True

We may not have expected that a false hypothesis implies anything and everything, but it will (hopefully) come to make sense later. Note that we say that p implies q is the truth value of $p \rightarrow q$ is true.

Here are some examples of true implications:

"Some bananas can be cooked while they are green $\rightarrow$ people can drink water."

"This sentence is written in kiSwahili $\rightarrow$ some bananas can be cooked while they are green."

"All cats in Tanzania are grey $\rightarrow$ this sentence is written in kiSwahili"

Note that though the sentences we make up this way do not make much sense in English, it still makes sense to speak of whether the implications are true or false. All 3 nonsensical sentences above are true implications. However, the following implication is false:

"People can drink water $\rightarrow$ all cats in Tanzania are grey".

In short, one may remember the truth table by the following rule: **Something true can only imply something true, but something false can imply anything.**

4 Bi-implications

The bi-implication or bi-conditional (also known as "if and only if") is the same as saying that both $p \rightarrow q$ and $q \rightarrow p$ are true, or that whenever you know one of them to be true, the other is also true. The truth table for $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ is shown below:

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \leftrightarrow q$
True	True	True	True	**True**	**True**
True	False	False	True	**False**	**False**
False	True	True	False	**False**	**False**
False	False	True	True	**True**	**True**

5 Tautologies

A tautology is a composite proposition for which all entries in the truth table are true. Similarly, a contradiction is a composite proposition for which all entries in the truth table are false.

Here are the truth tables for a few of the tautologies on page 16 of the chapter on logic:

p	$\neg p$	$p \vee \neg p$
True	False	**True**
False	True	**True**

p	$\neg p$	$(p \wedge \neg p)$	$\neg(p \wedge \neg p)$
True	False	False	**True**
False	True	False	**True**

p	q	$(p \vee q)$	$p \to q$
True	True	True	**True**
True	False	True	**True**
False	True	True	**True**
False	False	False	**True**

And finally the truth table for a contradiction:

p	$\neg p$	$p \wedge \neg p$
True	False	**False**
False	True	**False**

Remember where to find the list of tautologies for future reference. They will form the basis of how we argue later on, and thus it is very important to understand them and to know where to find them.

6 Logical Implication

We say that p logically implies q and write $p \Rightarrow q$ if the implication $p \to q$ is a tautology.

Recall that p and q are arbitrary propositions. We always just talk about the truth value of them, not the actual content. Here is one example of a logical implication:

p	q	$(p \vee q)$	$p \rightarrow (p \vee q)$
True	False	True	**True**
False	True	True	**True**
True	False	True	**True**
False	False	False	**True**

Thus, we can write $p \Rightarrow (p \vee q)$.

7 Logical Equivalence

Similarly, we say that p and q are logically equivalent if $p \leftrightarrow q$ is a tautology.

Note that we may also argue that 2 propositions are logically equivalent by showing that they have the same truth tables. Recall the truth table for $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$:

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \leftrightarrow q$
True	True	True	True	**True**	**True**
True	False	False	True	**False**	**False**
False	True	True	False	**False**	**False**
False	False	True	True	**True**	**True**

This shows that whenever the original propositions p and q have the same truth values, then $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ have the same truth values. and this the 2 composite propositions $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ are logically equivalent and we may write $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$.

Note that when 2 propositions are logically equivalent, we can substitute one for the other and choose the one we like best.