

Module 7: Quantifiers

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1 Introduction:

Logic is the basis of how we mathematicians argue. In order to be sure that an argument or a proof holds (is correct), we must follow the rules of (formal) logic in every step of the justification of it. The mathematical logic, or (formal) logic, doesn't look very similar to what we call logic in our everyday lives, but as we get more comfortable with it, we'll soon recognise that we use the same logical structures to justify arguments in spoken English - or kiSwahili.

This module corresponds to sections 1.6 and 1.7 in the chapter on logic.

2 Objectives:

At the end of this module the learner will be able to:

- Distinguish between the existential quantifier and the universal quantifier.
- Negate expressions with quantifiers.

3 The Existence Quantifier

Recall how in a previous module the proposition "All cats in Tanzania are grey" was negated by the existence of just one cat in Tanzania that isn't grey. This concept of the one case that breaks the rule (or less commonly confirms it) is so important in mathematics that it has its own symbol:

The existence quantifier: \exists . It is read "there exists".

So if I want to write about the cat which is not grey, I can write: \exists a cat in Tanzania such that this cat is not grey.

Note that we replace "such that" or "so that" with a colon or a vertical line. Thus the statement becomes:

\exists a cat in Tanzania | this cat is not grey.

or

\exists a cat in Tanzania: this cat is not grey.

Note that this does not mean that we can't have 500 cats which are not grey. As long as there is **at least one**, the statement is true.

The existence quantifier states that at least one part of the quantity it describes has the property that comes after the colon or vertical line.

4 The Universal Quantifier

If I don't want to negate my statement about the grey cats, I can use the universal quantifier: \forall . It is read "for all". The statement "All cats in Tanzania are grey" then becomes:

\forall cats in Tanzania | this cat is grey.

Note that the vertical line (or the colon) this time rather means something like "it is true that". If we can find just case that does not have the described quality (here: being grey), then the proposition is false.

The universal quantifier equips each and every part of the quantity it describes with the property that comes after the colon or vertical line.

5 Negating the Existence Quantifier

What is the opposite of the existence of at least one case of the described quality?

It is that none of the parts of the quantity in question has the described quality, or that it is true for all of them that they do not have the described quality.

In other words: Negating " $\exists x \in A|x$ has quality a " yields: " $\forall x \in A|x$ does not have quality a ".

6 Negating the Universal Quantifier

Similarly, if we negate the universal quantifier, we get the existence quantifier:

" $\neg\forall$ cats in Tanzania | this cat is grey" is " \exists " a cat in Tanzania | this cat is not grey.

7 Negating Composite Statements with Quantifiers

When we need to negate propositions with more than one quantifier, we do it successively.

If we want to negate the composite statement that in words reads: "For all types of bananas that grow in Tanzania it is true that there exists people who will eat them while they're still green", we need to put it in more mathematical terms first:

\forall types of bananas that grow in Tanzania | \exists people in Tanzania | this person eats these bananas green.

Now we have to break it down to statements:

$\underbrace{\forall \text{ types of bananas that grow in Tanzania}}_{\text{statement 1}} | \underbrace{\exists \text{ a person in Tanzania} | \text{this person eats these bananas green.}}_{\text{statement 2}}$

Note that we don't call them propositions as no truth value can be determined for a statement like: "For all types of bananas in Tanzania".

First, we negate statement 1: $\underbrace{\forall \text{ types of bananas that grow in Tanzania.}}_{\text{statement 1}}$

As explained in the previous section, we negate \forall by replacing it with \exists , so we get:

$\underbrace{\exists \text{ a type of banana that grows in Tanzania}}_{\neg \text{ statement 1}}$

Then we look at statement 2:

$\underbrace{\exists \text{ a person in Tanzania} | \text{this person eats these bananas green.}}_{\text{statement 2}}$

We notice that statement 2 is itself composite:

$$\underbrace{\exists \text{ a person in Tanzania}}_{\text{statement 2a}} \mid \underbrace{\text{this person eats these bananas green}}_{\text{statement 2b}}$$

Negating statement 2a, we get:

$$\underbrace{\forall \text{ people in Tanzania}}_{\neg \text{ statement 2a}}$$

And finally, negating statement 2b, we get:

$$\underbrace{\text{this person does not eat these bananas green}}_{\neg \text{ statement 2b}}$$

So the negation of statement 2 becomes:

$$\underbrace{\forall \text{ people in Tanzania} \mid \text{this person does not eat these bananas green}}_{\neg \text{ statement 2}}$$

Now, we put everything back together to get:

$$\underbrace{\exists \text{ a type of banana that grows in Tanzania}}_{\neg \text{ statement 1}} \mid \underbrace{\forall \text{ people in Tanzania} \mid \text{this person does not eat these bananas green}}_{\neg \text{ statement 2}}$$

To put it in human words, we have that there exists a type of bananas in Tanzania that nobody eats while they're green.

Note that it is always a good idea to check if the statement you have negated makes sense in plain words. This is not always easy, but it can enable you to catch errors. Similarly, you can check if your arguments against something make sense by breaking them into mathematical statements and negating them.

8 Negating Various Mathematic Symbols

Of course we can negate other mathematical symbols than \exists and \forall . Here's a table of mathematical symbols and what they are when you negate them:

Symbol	\neg Symbol
\in	\notin
\geq	$<$
\leq	$>$
$=$	\neq
\vee	\wedge
\wedge	\vee
\subseteq	$\not\subseteq$
\exists	\forall
\forall	\exists

Note that for some symbols, you need to take a longer route because you can't just negate them. Here's a table of those that we have met this far:

Expression	Change to:	Negate that:
$p \rightarrow q$	$(\neg p \vee q)$	$(p \wedge \neg q)$
$p \leftrightarrow q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$	$(\neg p \vee \neg q) \wedge (p \vee q)$

Note that when you negate an expression such as $(p \vee q)$, you need to negate all the terms of it to get $(\neg p \wedge \neg q)$.