Module 8: The Logic and the Methods of Proof

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1 Introduction:

Logic is the basis of how we mathematicians argue. In order to be sure that an argument or a proof holds (is correct), we must follow the rules of (formal) logic in every step of the justification of it. The mathematical logic, or (formal) logic, doesn't look very similar to what we call logic in our everyday lives, but as we get more comfortable with it, we'll soon recognise that we use the same logical structures to justify arguments in spoken English - or kiSwahili.

This module corresponds to sections 1.5 and 1.8 in the chapter on logic.

2 Objectives:

At the end of this module the learner will be able to:

- Use the logical calculus.

- Recognise how tautologies are used in mathematical argumentation.

- Distinguish between the different methods of proof.

3 The Logical Calculus

In order to use the logical calculus, we need to be rather familiar with the list of tautologies provided in the table on page 16 in the chapter of logic.

Basically, the logical calculus is about recognising collections of expressions and substituting them for the expressions they're leading to. In order to do this, we must of course be able to find these collections of expressions and then know what we can replace them with.

It is customary to list your expressions vertically. Let's take an easy example first. Given the expressions p and $(p \to q)$, we get:

$$\begin{array}{l} p \\ (p \to q) \\ \hline ? \end{array}$$

We recognise that $p \wedge (p \to q) \to q$ is a tautology, so we can replace the 2 original expressions p and $(p \to q)$ with q. Normally, we write this as:

p
$(p \to q)$
$\overline{\quad\quad\quad\quad}$
q

And we read it as "p and $(p \to q)$ leads to q" or "q is a logical consequence of p and $(p \to q)$". Note that this really means that we assume that p and $(p \to q)$ are true.

To take a slightly longer example, suppose that we have that:

$(p \wedge q) \vee (\neg p \wedge \neg q)$
q
$\overline{\quad}$
?

We recognise that $(p \wedge q) \vee (\neg p \wedge \neg q)$ is equivalent to $p \leftrightarrow q$, and that $p \leftrightarrow q$ implies $q \to p$, so we add it before the conclusion:

$(p \wedge q) \vee (\neg p \wedge \neg q)$
q
$(q \to p)$
$\overline{\quad\quad\quad\quad}$
?

Now we know that $q \wedge (q \to p) \to p$, so we get:

$(p \wedge q) \vee (\neg p \wedge \neg q)$
q
$(q \to p)$
$\overline{\quad\quad\quad\quad}$
p

The logical calculus helps us reduce a large amount of propositions to something that's intelligible.

4 Methods of Proof

Here is a table of things to prove and methods to prove them. Recall from previous modules that a contradiction is a (composite) proposition that is always false no matter what the original propositions were. In the table below, c denotes a contradiction:

Proposition	Method 1	Method 2	Method 3	Method 4
$p \Rightarrow q$ Implication	$p \Rightarrow q$ Direct Proof	$\neg q \Rightarrow \neg p$ Contrapositive	$(p \wedge \neg q) \Rightarrow c$ Indirect Proof	$(r \Rightarrow q) \wedge (p \Rightarrow r)$ Proof by Substitution
$p \Leftrightarrow q$ Bi-implication	$p \Leftrightarrow q$ Direct Proof	$\neg p \Leftrightarrow \neg q$ Contrapositive	$(p \Rightarrow q) \wedge (q \Rightarrow p)$ Breaking into Parts	$(r \Leftrightarrow q) \wedge (p \Leftrightarrow r)$ Proof by Substitution

Note that if we choose the proof by breaking into parts method, we can go to the first line and choose any of the methods of proof for a single implication, and we can even prove the 2 implications in 2 different ways.

To give an example, recall that the integers, $\mathbb{Z}$, is the set $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, and that the natural numbers, $\mathbb{N}$, is the set $\{x \in \mathbb{Z} | x > 0\} = \{1, 2, 3, 4, \ldots\}$. We want to prove the following:

Theorem 1: $\forall n \in \mathbb{Z} : n^3 > 0 \Leftrightarrow n > 0$

The theorem states that for all integers n, we have that n^3 is greater than 0 if and only if n itself is greater than 0. We'll prove it by breaking it into parts:

Part 1: $n > 0 \Rightarrow n^3 > 0$. Assume that $n > 0$. $n^3 = n \cdot n \cdot n$, and as $n > 0$, we haven't multiplied by anything negative, so n^3 must be greater than 0.

This was a direct proof.

Part 2: $n^3 > 0 \Rightarrow n > 0$: Assume that $n < 0$. As $n^3 = n \cdot n^2$ and we know that for all integers m, $m^2 > 0$, we have that $n \cdot n^2 < 0$ as we assumed that $n < 0$ and something negative multiplied by something positive yields a negative number. So $n^3 < 0$.

This was a proof by contraposition $((p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p))$.

There's a bit more to be said about proof by substitution. We may look at the table of tautologies to find any form r that will suit our purposes. This takes some practise, though.