# VISTOOMA, VIsualisation TOOl for MAth. Module 3: Proof by Induction

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## Contents



## 1 Module 3: Proof by Induction

The proof by induction is an example of exploiting a unique property of the elements that define a set to prove far-reaching propositions for all elements of that set. In this case, it is the set of natural numbers, $\mathbb{N}$, and the property we use is the fact that you can get from any element of $\mathbb{N}$ to the next by adding one.

However, this disciplin is one that challenges students very much, as it is one of their first encounters with mathematical assumptions and the rules of mathematical argumentation. A proof by induction consists of 3 steps: The start, the induction hypothesis or induction assumption, and the conclusion.

An example of a proof by induction is the following: We wish to prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for any natural number $n$.

**START:** Check that the identity holds for very small values of $n$. Checking it for $n = 1$ or $n = 2$ suffices. Note that students tend to dislike checking it for $n = 1$ as the concept of a sum of one number can be difficult to come to terms with:

$n = 1 : 1 = \frac{1\times(1+1)}{2} = \frac{2}{2} = 1$
$n = 2 : 1 + 2 = 3 = \frac{2(2+1)}{2} = \frac{6}{2} = 3$

**INDUCTION HYPOTHESIS:** Since the identity holds for small values of $n$, we assume that it holds for all natural numbers $k$ such that $k \leq m$, where $m$ denotes some arbitrary but set natural number. If we can show that whenever the identity holds for $m$, then it holds for $m+1$, we have in fact shown that it holds for all natural numbers since $m$ is arbitrary and we can generate any natural number by adding one to the natural number that is one less (or equivalently, startig from one and adding one a sufficient number of times).

It is this step that is the greatest challenge for students, both in terms of making the correct induction hypothesis and in terms of using the induction hypothesis correctly in the last part of the proof.

**CONCLUSION:** As we have assumed that the identity holds for all $k \leq m$, we need to check what happens for $m+1$:

$\sum_{i=1}^{m+1} i =$
$\sum_{i=1}^{m} i + (m+1) =$ (Splitting the sum)
$\frac{m(m+1)}{2} + (m+1) =$ (Using the induction hypothesis)
$\frac{m^2+m}{2} + \frac{2m+2}{2} =$
$\frac{m^2+3m+2}{2} =$
$\frac{(m+1)(m+2)}{2}$

The pitfalls here are splitting the sum so that you may apply the induction hypothesis, and identifying the resulting formula as the original formula with $m$ replaced by $m+1$. For some students, correctly reading a formula such as $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ is also a challenge.

## 1.1 Software Requirements for Module 3: Proof by Induction

### 1.1.1 Software Requirements for Reading Formulas

This section aims at assisting the student in understanding formulas such as $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. The student can either choose to increment the formula by 1 or to roll it out.

Incrementing $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ by one yields $\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$ (which you can then increment by one), and rolling it out yields $\sum_{i=1}^{n} i = 1 + 2 + 3 + 4 + 5 + 6 + \cdots + n$, or if you specify e.g. $n = 5, 1 + 2 + 3 + 4 + 5 = 15$.

Vistooma needs the software that can add one to the number (placeholder) $n$ in the formula, but in a way so that $n+1$ can be replaced by $n+2$ and so on, and the software to roll out sums. This routine would be very similar to the function called $p(n)$ below.

#### 1.1.2 Software Requirements for Proof by Induction

As this module is only about proofs by induction, the software required is a "proof by induction wizard" (see "User Manual") along with routines to do some symbolic manipulation and a function that can take a typed-in input (e.g. the input $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and generate the function $p(n)$, which applied to a number $k$ returns the number $p(k)$, and applied to a string $m$ returns either the string $\sum_{i=1}^{m} i$, $\frac{m(m+1)}{2}$ or $\sum_{i=1}^{m} = \frac{m(m+1)}{2}$ ).

The routines need to be able to recognise the *shape* of an argument as the student isn't obliged to use the same letters in the calculations as have been used in the examples.

#### 1.1.3 Software Requirements for Menus

This Module needs a menu which will let the student choose between doing Proof by Induction or working with formulas. The menu item **Formulas** furthermore needs to allow the student to choose between rolling out formulas and incrementing them.

#### 1.1.4 Software Requirements for Worked Examples

This requires a simple database of worked examples of proofs by induction, along with a random generator.

### 1.2 "User Manual": Reading Formulas

In this section, you can learn how to read formulas, either by rolling them out to understand what they really mean, or to learn how to add one to the counter $n$. If you choose **Formulas** and then **Roll Out Formula**, you'll get a dialogue box such as the following:

| Formula | Type in the rolled-out formula | Specify $n$ | Result |
|---|---|---|---|
| $\ldots$ | $= \ldots, \ldots$ times | $n =$ | $= \ldots$ |

Which you then fill in, here shown with $\sum_{i=1}^{n} n$:

| Formula | Type in the rolled-out formula | Specify $n$ | Result |
|---|---|---|---|
| $\sum_{i=1}^{n} n$ | $= n + n + \cdots + n$, $n$ times | $n = 5$ | $\sum_{i=1}^{5} 5 = 5 + 5 + 5 + 5 + 5 = 25$ |

Vistooma will tell you if you make a mistake.

If you choose **Formulas** and then **Increment Formula**, you'll get a dialogue box such as the following:

| Formula | Incremented by One |
|---|---|
| ... | ... |

Which you then fill in, here shown with $(ab)^n = a^n b^n$:

| Formula | Incremented by One |
|---|---|
| $(ab)^n = a^n b^n$ | $(ab)^{n+1} = a^{n+1} b^{n+1}$ |

Again, Vistooma will tell you if you make a mistake.

## 1.3 "User Manual": Proof by Induction

Vistooma assists you with proofs by inductiction by using a wizard to guide you every step along the way. Choosing **Proof by Induction** will open the wizard. It contains the following steps:

| You | Vistooma | Examples |
|---|---|---|
| Type in the identity you wish to prove using the charmap table | generates a function $p(n)$ that will take a natural number as argument and do the desired operations with it | $p(n) = \sum_{i=1}^{n} i$ |
| Check if the identity holds for $n = 1$ or $n = 2$ | applies $p$ to the typed-in value of $n$ and compares with your result, returning an error message if you made a calculation error | $p(1) = 1$, $p(2) = 3$ |
| Type in your induction hypothesis | generates a string saying: **"Assume that for any natural number** $k \leq m$ : $p(k)$", where $p(k)$ is expressed in the symbolic form. If your induction hypothesis is different from this, Vistooma generates an error message and prints the correct one | **Assume that for any natural number** $k \leq m : \sum_{i=1}^{k} = \frac{k(k+1)}{2}$ |
| Type in the sum, product or difference for $n = m+1$ | generates the symbolic form of $p(m+1)$ and returns an error message if your input is different | **"You need to start this part of the proof with an expression such as** $\sum_{i=1}^{m+1} i$**"** |
| Split the sum, product or difference into appropriate parts | generates a symbolic expression on the form $p(m)+(m+1)$ and returns an error message if your input doesn't match | **"The correct way to proceed is to look at** $\sum_{i=1}^{m+1} i = \sum_{i=1}^{m} +(m+1)$ **so that you can apply the induction hypothesis to** $\sum_{i=1}^{m} i$**"** |
| Type in the induction hypothesis in stead of the sum expression | generates the alternative formulation of $p(m)$ in the symbolic form | $p(m) = \frac{m(m+1)}{2}$ |
| Do the calculations to arrive at the expression for $n = m+1$ | generates the alternative formulation of $p(m+1)$ in the symbolic form and compares with your result, returning an error message if you do not arrive at the same result | **"Please try again.** $\sum_{i=1}^{m+1} i = \frac{(m+1)(m+2)}{2}$**, which proves the identity."** |

In case the identity you're trying to prove is wrong, Vistooma will return an error message with an example of where it goes wrong, i.g. "It is not true that $\sum_{i=1}^{n}(-1)^{i+1}i = (-1)^{i+1}2$ as $\sum_{i=1}^{5}(-1)^{i+1}i = 3$".

If it is not possible to find a counter example within a reasonably small subset of $\mathbb{N}$, Vistooma will just tell you that the proof by induction failed.

## 1.4 "User Manual": Worked Examples

Vistooma provides a selection of worked examples for each module. Here, you can randomly generate a worked example to get a feel for the functionalities or to practise your understanding by looking at examples.