VISTOOMA, VIsualisation TOOl for MAth. Module 7: Methods of Proof

Signe Hermann

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1 Module 7: Methods of Proof

I'll start this Module with a quote from one of my lecturers at the University of Copenhagen: **"It's always like that with mathematical proofs: We set out to prove one thing, then we prove something completely different, and then everybody is happy!"**

Of course there are rules that tell us which "completely different things" we can prove instead of what we set out to prove in the first place, and they're based on the list of tautologies used in module 6. In a sense, it's just a change of outlook or angle of approach, a simple example of which is the following: We have the logical equivalences

$(p \to q) \leftrightarrow (\neg q \to \neg p) \leftrightarrow ((p \wedge \neg q) \to c)$ (where c denotes any contradiction),

so mathematically speaking, for propositions p and q, proving $p \to q$ is the same as proving $\neg q \to \neg p$ or $(p \wedge \neg q) \to c$, and it is only a matter of disposition and creativity which one you prefer, though the proofs themselves might look somewhat dissimmilar to the untrained eye:

Let the statement $p = n, m \in \mathbb{Z}$ are even, and statement $q = n + m$ is even. We want to show that p implies q.

DIRECT PROOF: This is $p \to q$. Assume that n, m are even. Then we can find s, t so that $n = 2s$ and $m = 2t$. Now, $n + m = 2s + 2t = 2(s + t)$. As s, t are integers, $(s + t)$ is an integer, say $s + t = k$, and thus $n + m = 2(s + t) = 2k$, which is an even integer.

INDIRECT PROOF / PROOF BY CONTRAPOSITION: This is $\neg q \to \neg p$. Assume that $n + m$ is odd. Now, we can find j so that $n + m = 2j + 1$, or $n = 2j - m + 1$. If we assume that either n or m is odd, we're done with the proof, so assume both of them are even. Now, we can find s, t so that $n = 2s$ and $m = 2t$. Inserting this in the formula, we get $2s = 2j - 2t + 1 = 2s = 2(j - t) + 1$. As j, t are integers, $(j - t) = u$ is an integer, and thus we have that $2s = 2u + 1$. This is clearly impossible, so either n or m must be odd.

Note that the indirect proof required the use of another logical equivalence, $(x \to y) \leftrightarrow (x \wedge \neg y) \to c$, which is also used below:

PROOF BY REDUCTIO AD ABSURDUM This is $(p \wedge \neg q) \to c$. Assume that n, m are even and that $n + m$ is odd. We can find s, t, u so that $n = 2s, m + 2t$ and $n + m = 2u + 1$. Now, we have that $n + m = 2s + 2t = 2(s + t) = 2u + 1$, which is clearly impossible as $s + t$ is an integer.

Understanding that the various methods of proof are freely interchangeable as well as switching from one angle of approach to another is a big challenge for most students. Whereas the form $p \to q$ is generally easy enough to come to terms with, most students automatically try to prove $\neg p \to \neg q$ in stead of $\neg q \to \neg p$, but $\neg p \to \neg q$ and $p \to q$ are not logically equivalent, and thus they have not proven what they set out to prove.

Note that the atomic statements used in the methods of proof are actually different from what we'd normally accept as mathematical propositions. This is a trade-off due to the fact that the statements we want to prove have to be applicable to all integers, and thus we have to make "propositions" with unknowns.

1.1 Software Requirements for Module 7: Methods of Proof

1.1.1 Software requirements for Mathematical Propositions

This is a recycling of the functionalities from Module 5: Truth Tables.

1.1.2 Software Requirements for Methods of Proof

In order to train the student in the various methods of proof, Vistooma has a database of very simple theorems coupled with the different ways of proving them. The student must first identify the statements involved, and then read the proof presented and finally choose the correct symbolic analogue of the proof.

From the above example, the theorem would be:

The sum of 2 even integers is an even integer.

The students will then be asked to identify the statements involved from a multiple choice dialogue box such as

Identify the statements in the theorem	Select one or more
n is an even integer	○
$n+m$ is an odd integer	○
n, m are even integers	○
n, m are odd integers	○
$n+m$ is an even integer	○

If the student chooses a wrong proposition, such as

Identify the statements in the theorem	Select one or more
n is an even integer	○
$n+m$ is an odd integer	$\otimes$
n, m are even integers	$\otimes$
n, m are odd integers	○
$n+m$ is an even integer	○

Vistooma returns an error message such as "Please try again. Are you sure that "$n+m$ is an odd integer" is a statement used in the theorem **The sum of 2 even integers is an even integer?**".

If the students correctly identifies the relevant statements, such as in

Identify the statements in the theorem	Select one or more
n is an even integer	○
$n+m$ is an odd integer	○
n, m are even integers	$\otimes$
n, m are odd integers	○
$n+m$ is an even integer	$\otimes$

Vistooma will return the message: "That is correct. The statements used in the theorem **The sum of 2 even integers is an even integer**" are "$p = n, m$ are even integers" and "$q = m + n$ is an even integer."

Now, Vistooma will randomly generate a proof of the theorem from the database along with another dialogue box, such as

The statements used are:

$p = n, m$ are even integers
$q = m + n$ is an even integer.

PROOF: Assume that n, m are even and that $n + m$ is odd. We can find s, t, u so that $n = 2s, m + 2t$ and $n + m = 2u + 1$. Now, we have that $n + m = 2s + 2t = 2(s + t) = 2u + 1$, which is clearly impossible as $s + t$ is an integer.

Method of Proof	Select One	Name of Method
$p \to q$	○	Direct Proof
$\neg q \to \neg p$	○	Indirect Proof
$(p \wedge \neg q) \to c$	○	Reductio ad Absurdum

If the student makes in incorrect choice, such as choosing $\neg q \to \neg p$ (Indirect Proof), Vistooma will generate an error message such as: "Please try again. The indirect proof of the theorem **The sum of two even integers is an even integer** runs as follows:

The statements used are:

$p = n, m$ are even integers. The negation is "$\neg p =$ at least one of the 2 integers n, m is odd"
$q = m + n$ is an even integer. The negation is "$\neg q = n + m$ is an odd integer"

The indirect proof is $\neg q \to \neg p$, or **If the sum of two integers is an odd integers, then at least one of them must be odd:** Assume that $n + m$ is odd. Now, we can find j so that $n + m = 2j + 1$, or $n = 2j - m + 1$. If we assume that either n or m is odd, we're done with the proof, so assume both of them are even. Now, we can find s, t so that $n = 2s$ and $m = 2t$. Inserting this in the formula, we get $2s = 2j - 2t + 1 = 2s = 2(j - t) + 1$. As j, t are integers, $(j - t) = u$ is an integer, and thus we have that $2s = 2u + 1$. This is clearly impossible, so either n or m must be odd."

If the student makes a correct choice, Vistooma returns the following message: "That is correct. The method of proof used here is Reductio ad Absurdum".

The software required here is a database of theorems coupled with all the methods of proof, a random generator which will choose a theorem and one of the methods of proof, and a routine to dynamically create the multiple choice dialogue boxes needed to first identify the statements or generate error messages if a wrong statement is used, and then the method of proof after it has been presented. A routine to generate the right error message is also needed, that is to present the paedagogical explanation why the method of proof chosen is not correct.

1.2 "User Manual": Mathematical Propositions

See Module 5.

1.3 "User Manual": Methods of Proof

Vistooma's Module 7 is generally a database of theorems and methods of proof, and when you choose to get a theorem, you will first be asked to select the statements involved:

THEOREM: The sum of 2 even integers is an even integer.

Identify the statements in the theorem	Select one or more
n is an even integer	◯
$n + m$ is an odd integer	◯
n, m are even integers	◯
n, m are odd integers	◯
$n + m$ is an even integer	◯

When you have identified the correct statements, Visatooma will present you with a proof of the theorem and ask you to identify the method of proof used:

The statements used are:

$p = n, m$ are even integers.
$q = n + m$ is an even integer.

PROOF: Assume that n, m are even. Then we can find s, t so that $n = 2s$ and $m = 2t$. Now, $n+m = 2s+2t = 2(s+t)$. As s, t are integers, $(s + t)$ is an integer, say $s + t = k$, and thus $n + m = 2(s + t) = 2k$, which is an even integer.

Method of Proof	Select One	Name of Method
$p \to q$	◯	Direct Proof
$\neg q \to \neg p$	◯	Indirect Proof
$(p \wedge \neg q) \to c$	◯	Reductio ad Absurdum

If you choose a wrong method of proof, Viastooma will show you what the proof by that method looks like so that you may understand why your choice of method of proof wasn't correct.

1.4 "User Manual": Worked Examples

Vistooma provides a selection of worked examples for each module. Here, you can randomly generate a theorem with all methods of proof presented.