Vistooma Textbook

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Contents

1 Module 1: Venn Diagrams and Set Algebra

1.1 Sets and the most common ways of writing them:

A set is a collection of items. The set is made up by these items. The items that make up a set determine it completely. Usually, but not always, the items that form the set will have some property in common. Sets can be finite (meaning that they consist of a finite number of items) or infinite (meaning that they consist of so many items that they can never be counted).

Common ways of writing a set is

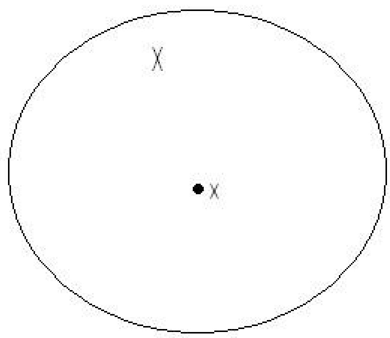
$\{1, 2, 3, 4, 5\}, A, B, C, \{2, 4, 6, 8, \dots\}, \{\dots, -2, -1, 0, 1, 2, \dots\}, \dots$

Some very important sets are the universal set, U, which in a sense describes the universe in which we work, "the set of everything", and $\emptyset$, the empty set. This means that $\emptyset$ is the set that is defined by having no items in it. Though defining these 2 sets may seem superfluous, they do actually play important roles in mathematics.

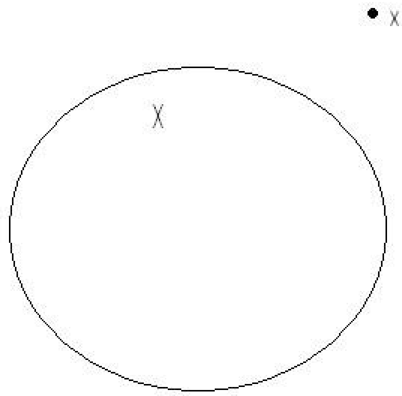
1.2 Elements of sets:

The items that a set is made up of are called elements. We speak of the elements of a set, and say that a set is completely determined by its elements. If a set is large, it is more feasible to describe it by some common property of its elements than by listing all the elements of it.

To say that some entity, say x, is an element of a set, say X, we write $x \in X$:



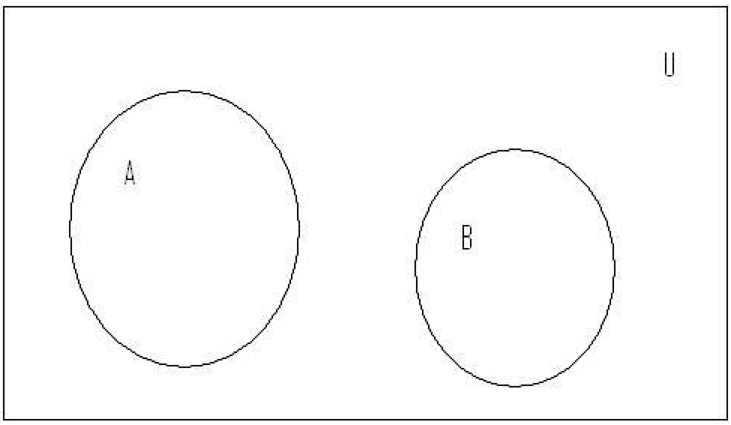
To say the opposite, that x is not an element of X, we write $x \notin X$:



A way of listing A by its elements is $A = \{x : x \text{ has a certain property}\}$ or $A = \{x|x \text{ has a certain property}\}$. The colon or vertical line is read as "which" or "such that", so A is the set of x which has a certain property, or the set of x such that x has a certain property.

1.3 Venn Diagrams:

A very graphically appealing way of understanding sets is by the so-called Venn diagrams. We make a large square box to illustrate U, the universal set, and make circles to illustrate the sets that weŠre interested in, say A and B:



This is to be interpreted as follows: The circle labelled "A" contains all the elements that the set A is determined by, and that the circle labelled

"B" contains all the elements that the set B is determined by. In this case, A and B have no elements in common.

1.4 Union and Intersection:

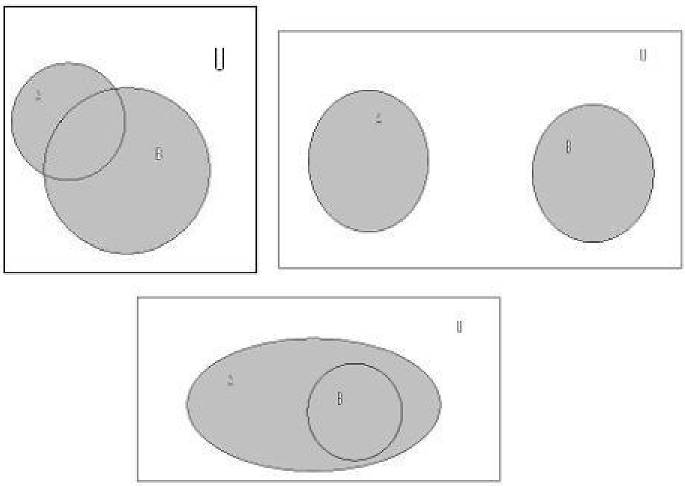
2 very common and very important set operations are the union of 2 sets and the intersection of 2 sets.

Using Venn diagrams to illustrate the concepts, we have that the union of 2 sets A and B are:

all the elements of A, together with

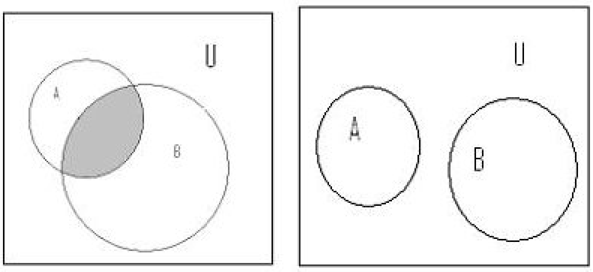
all the elements of B, together with

all the elements that are elements of A and B at the same time:



The above Venn diagrams all illustrate the union of 2 sets A and B. We write the union of 2 sets A and B as $A \cup B$.

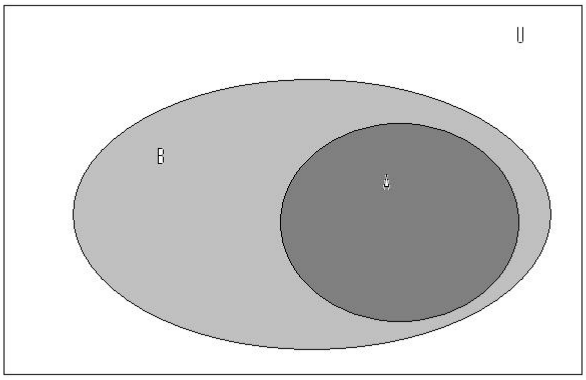
Similarly, the intersection of A and B, written $A \cap B$, denotes all the elements that are elements of A and B at the same time:



In case there are no elements that are elements of both A and B at the same time, we say that the intersection is empty, or that it equals the empty set, $\emptyset$. In symbols: $A \cap B = \emptyset$.

1.5 Subset and Superset

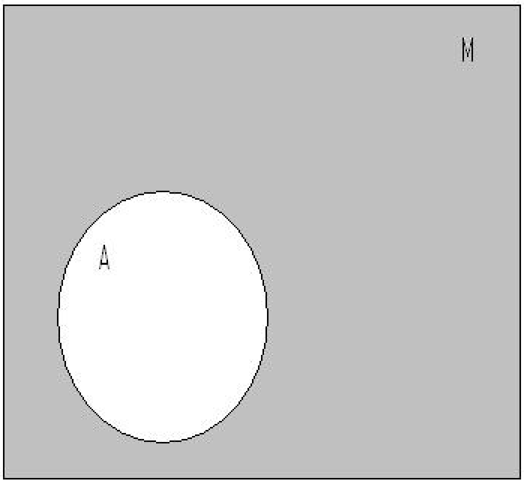
If all the elements in a set A are also elements of a set B, we say that A is a subset of B:



We write this $A \subset B$ or $A \subseteq B$. In this case, we say that B is a superset of A, which we write $B \supset A$ or $B \supseteq A$. The term superset is not commonly used, though.

1.6 Complement of a Set:

Using Venn diagrams to illustrate the complement of a set, we get that for a set A, the complement of A, written A^C, is the set of all the elements of the universal set U that are not elements of A:

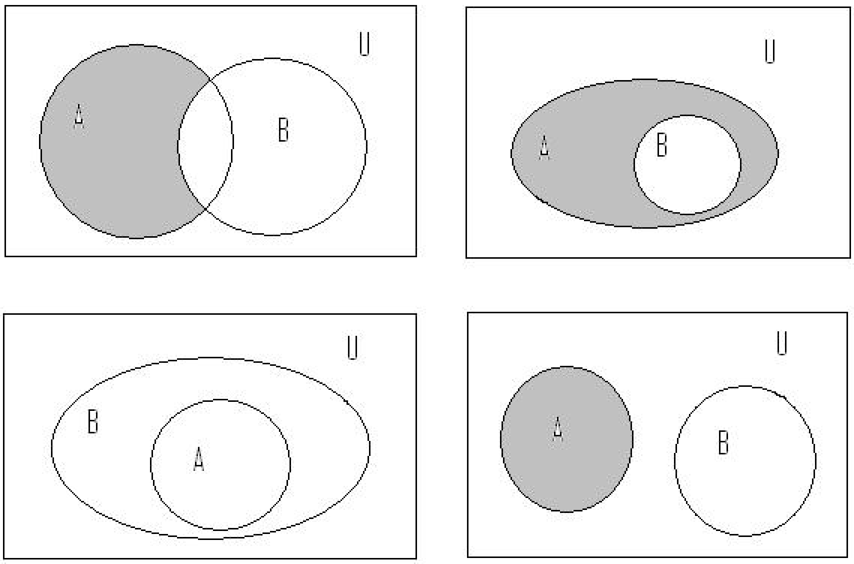


The shaded area is the complement of A, A^C.

1.7 Set Difference:

The set difference between 2 sets A and B is written $A \setminus B$ and denotes the elements of U that are elements of A, but are not elements of B.

Using Venn diagrams to illustrate the concept:

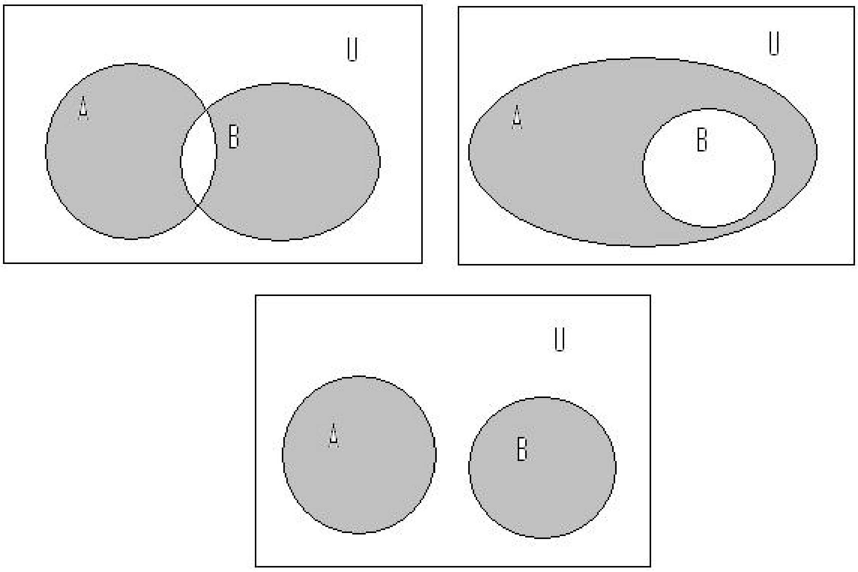


All 4 Venn diagrams illustrate set differences. Note that when A is completely conrtained in B, the set difference $A \setminus B$ is the empty set, $\emptyset$, and when A and B have no elements in common, the set difference $A \setminus B$ is simply A. In terms of complements, we see that $A^C = U \setminus A$.

Note, too, that $A \setminus B$ is not the same as $B \setminus A$.

1.8 Symmetric Difference:

Recall from last module that the union of 2 sets A and B consists of all the elements of the universal set U that are elements of A or B or both. The symmetric difference between 2 sets A and B consists of all the elements of $A \setminus B$ and all the elements of $B \setminus A$, or in symbols: $(A \setminus B) \cup (B \setminus A)$. We write the symmetric difference between A and B as $A \triangle B$ or $A \oplus B$.



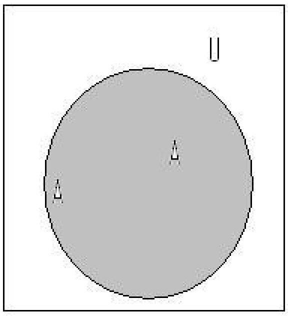
Note that if all the elements of B are contained in A, the symmetric difference is the same as $A \setminus B$, and if A and B have no elements in common, the symmetric difference is the same as $A \cup B$.

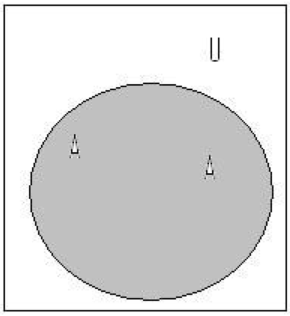
Note, too, that $A \triangle B = B \triangle A$, which is why it is called a symmetric difference.

1.9 The Algebra of Sets:

Recall from last module that the intersection of 2 sets A and B are the elements of the universal set U which are elements of both A and B.

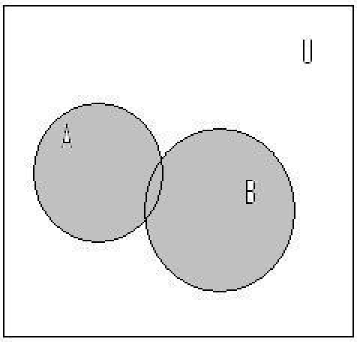
As noted with a lot of the set operations, the order in which you apply them is very important. The operations for which the order does not matter are the following:

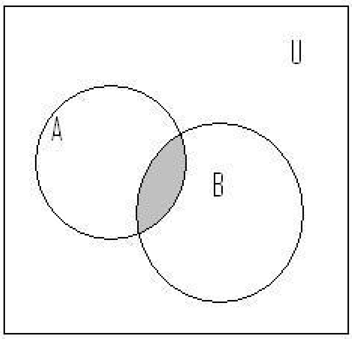




$A \cup A = A$ and $A \cap A = A$
Idempotent Laws

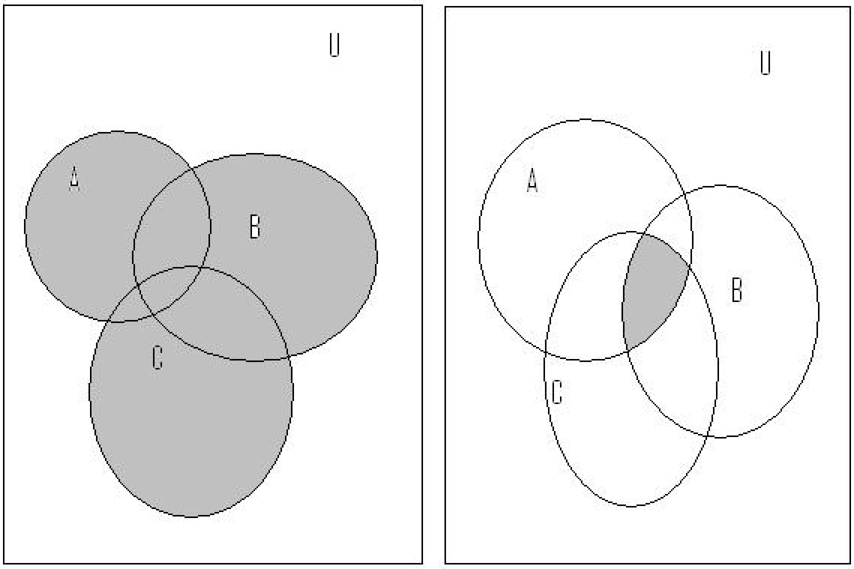
That an operation is commutative means that you can swap the order of the operands:





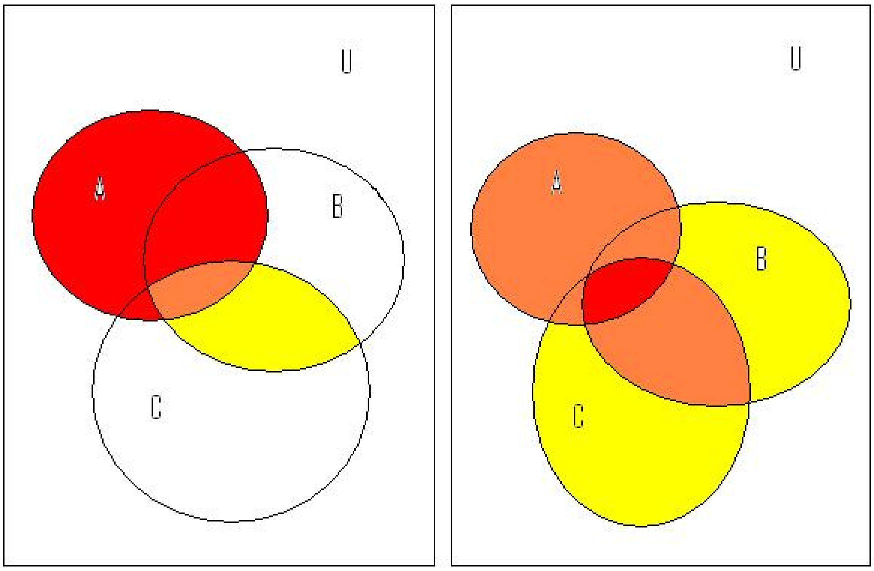
$A \cup B = B \cup A$ and $A \cap B = B \cap A$
Commutative Laws

That an operation is associative means that when you apply it to more than 2 operands, the order in which you apply it doesn't matter:

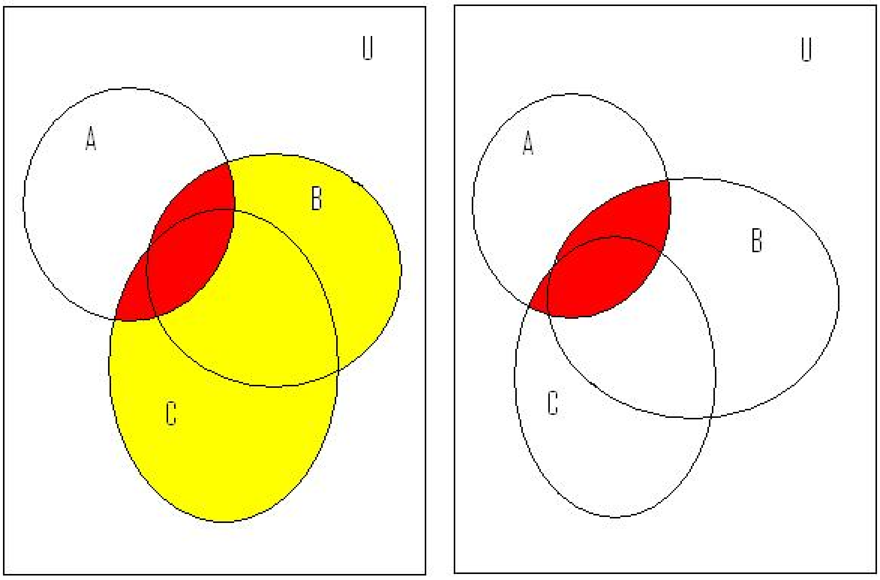


$A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$
Associative Laws

The Distributive Laws tell us how the operations are distributed with respect to each other:



$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Distributive Law



$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Distributive Law

We also have the following identities, which are difficult to draw as Venn diagrams, but which are very obvious once you think about them:

$A \cup \emptyset = A$ (Identity Law)

$A \cap U = A$ (Identity Law)

$A \cup U = U$ (Identity Law)

$A \cap \emptyset = \emptyset$ (Identity Law)

$(A^C)^C = A$ (Involution Law)

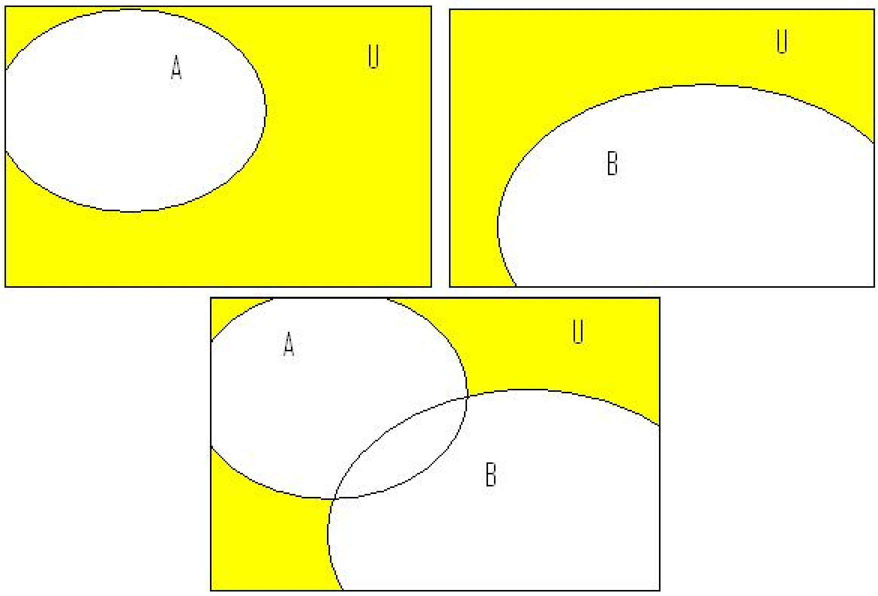
$A \cup A^C = U$ (Complement Law)

$A \cap A^C = \emptyset$ (Complement Law)

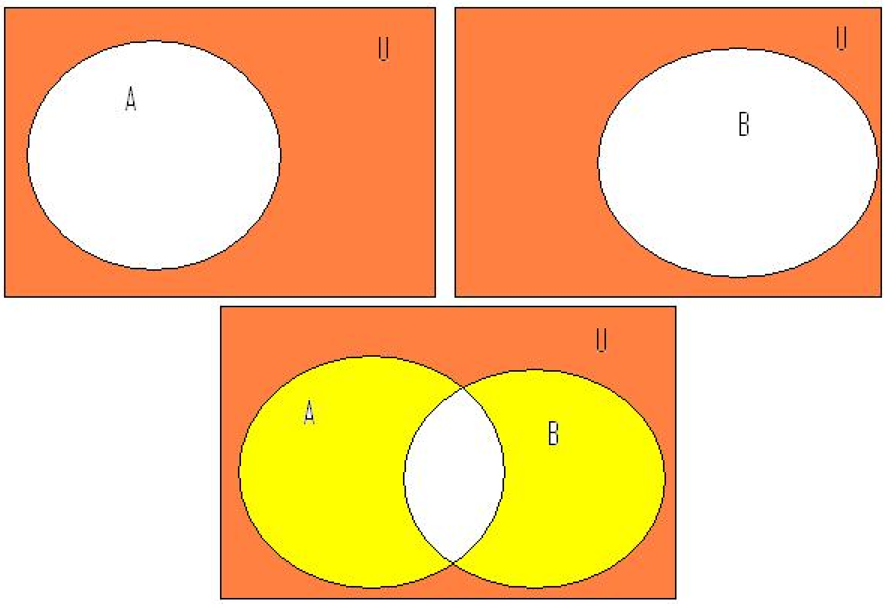
$U^C = \emptyset$ (Complement Law)

$\emptyset^C = U$

Finally, we have De Morgans Laws, which are very important in (formal) logic. Here, we meet them in their set theoretic interpretation:



$(A \cup B)^C = A^C \cap B^C$
De Morgan's Law



$(A \cup B)^C = A^C \cap B^C$
De Morgan's Law

1.10 Duality

Duality is a core concept of set theory. We get the dual of a statement by replacing the operators pairwise as follows:

$\cup$ is replaced by $\cap$

$\cap$ is replaced by $\cup$

U is replaced by $\emptyset$

$\emptyset$ is replaced by U

The good thing about duality is that if you have an identity, the dual of that identity is automatically an identity, too.

To get the dual in the exemple of the book, we do as follows, evaluating the parentheses first:

START: $(U \cap A) \cup (B \cap A)$

$U \to \emptyset, \cap \to \cup, A = A$ and $B = B, \cap \to \cup, A = A$, so we get:

$(U \cap A) \to (\emptyset \cup A)$ and $(B \cap A) \to (B \cup A)$

Having dealt with the parentheses, we get $\cup \to \cap$, so the dual becomes:

$(\emptyset \cup A) \cap (B \cup A)$.

Using the laws of set algebra from the last section, we see that:

$(\emptyset \cup A) = A$ and $A \cap (B \cup A) = A$, or with the dual expression:

$(U \cap A) = A$, and $A \cup (B \cap A) = A$. For the last identity we have used that all elements of $A \cap B$ are elements of A, so that we do not add anything that wasn't an element of A to begin with.

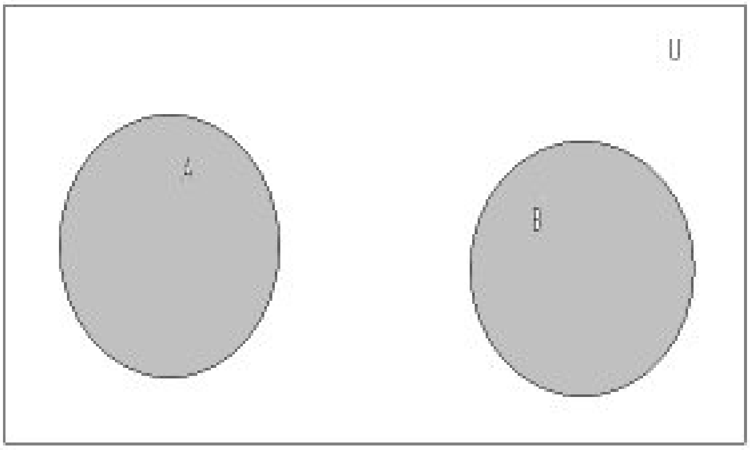
2 Module 2: Venn Diagrams, Logical Argumentation and Counting Principles

2.1 Finite Sets:

A set is called finite if it has a finite amount of elements. A finite amount means that if you count them, you will finish before the end of time. If a set has a finite amount of elements, counting these elements make sense, but if it hasn't, you will never finish your task if you set out to count them.

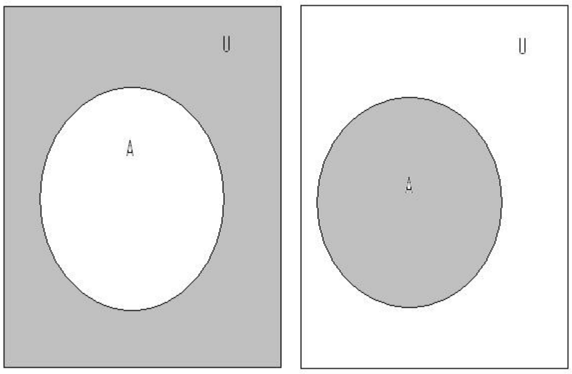
2.2 Counting the Elements of Disjoint Sets:

That 2 sets are disjoint means that they have no elements in common:



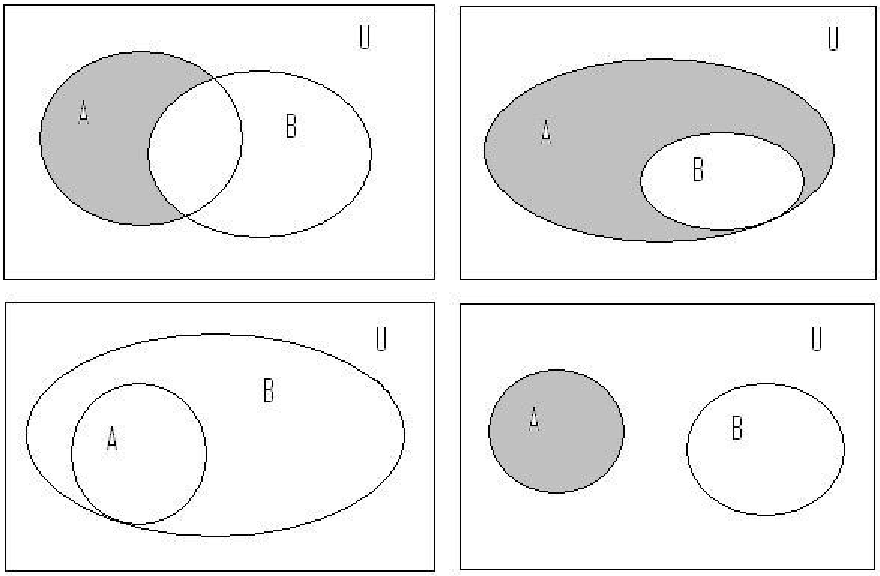
In this case, we simply count the elements of one set and add that number to the number of elements in the other set. Denoting the number of elements of a set A by $n(A)$, we get that as long as A and B are disjoint sets, then $n(A \cup B) = n(A) + n(B)$.

Similarly, if we want to count the elements of the universal set U, assuming that it is finite, we get that $n(U) = n(A) + n(A^C)$:



From this, we can easily deduct that $n(A^C) = n(U) - n(A)$ and $n(A) = n(U) - n(A^C)$.

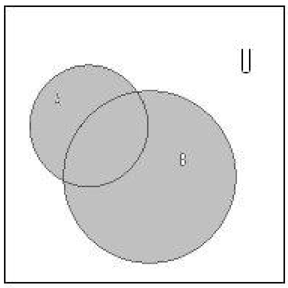
Further, we have that for 2 finite sets A and B, $n(A \setminus B) = n(B) - n(A \cap B)$:



Note that if A is completely contained in B, there are no elements in $A \setminus B$, and if A and B are disjoint, $n(A \setminus B) = n(A)$.

2.3 Counting Elements of Sets that are not Disjoint:

If we want to count the elements of sets that are not disjoint, we have to be a little more clever:

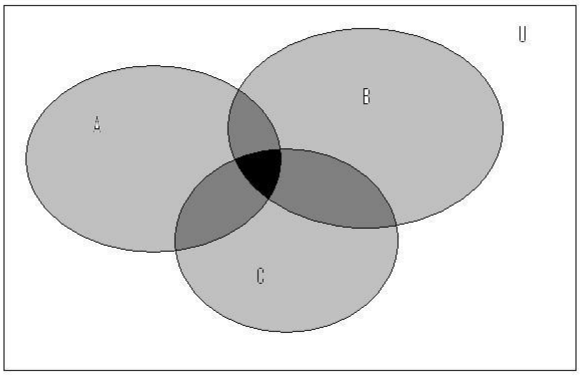


If we just use the same principle as with disjoint sets, we will have counted the elements that are in both A and B twice, so that approach doesn't work. What we do is to count them twice and them subtract them once:

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Note that this is the same formula as with the disjoint sets, as the intersection of disjoint sets is empty by definition and thus contains no elements.

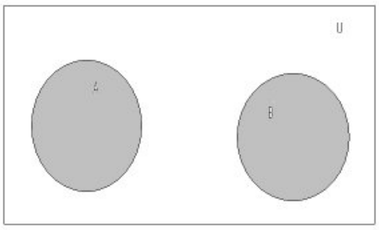
Similarly, if we have 3 sets with a non-empty intersection, we add the elements of each set, subtract the elements that we have added twice, but now we will have subtracted the elements that are in all 3 sets one time too many, so we add those:



$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

2.4 Power Sets and Classes of Sets

Recall from the first module that the statement $A \subseteq B$ (A is a subset of B) means that if you have an element of A, it is also an elements of B:



$$A_5 \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1 \subseteq A$$

Note that the relation $A_5 \subseteq A_4$ automatically means that A_5 is a subset of all the sets that A_4 is a subset of: $A_5 \subseteq A_3$, $A_5 \subseteq A2$, $A_5 \subseteq A_1$ and $A_5 \subseteq A$. When a relation has the character, we say that it is transitive.

The power set of a set is the set of all subsets of that set. As a matter of convention, the empty set, $\emptyset$ is always an element pf the powerset of a set. We write the powerset of a set A as $\mathcal{P}(A)$. In our case above (the Venn diagram), the power set of A has around 30 elements though we only have the set A and it's 5 subsets. This is because we must also consider all possible combinations of subsets of A, for instance $(A \setminus A1) \cup (A2 \setminus A3) \cap (A4\ A5)$

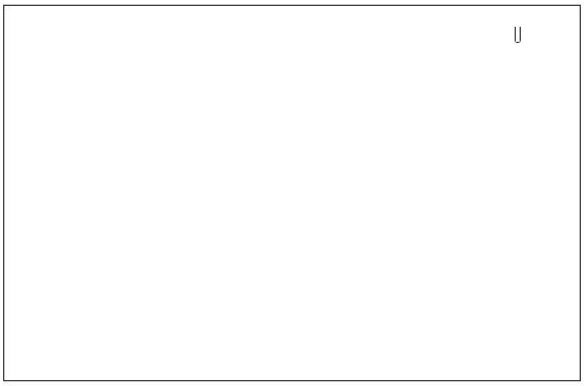
will be in it.

The reason why we include the empty set and the whole set is not apparent here, but it will allow us to perform any set operation on A and always get a result that is an element of $\mathcal{P}(A)$.

Note that is is perfectly possible for the elements of a set to be sets. However, we may speak of a class of sets or a collection of sets instead of speaking of a set of sets in order to avoid confusion.

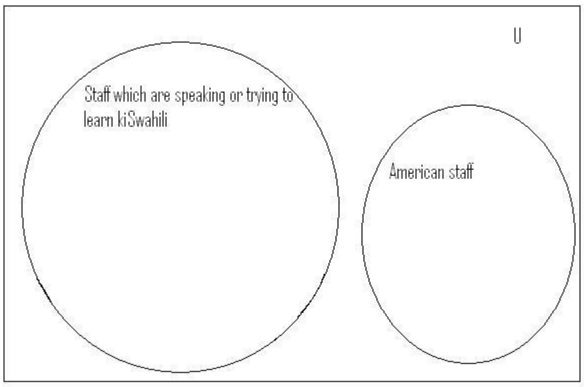
2.5 Using Venn Diagrams to Justify Arguments:

The task is to imagine some (undescribed) universal set that everything you argue about is con- tained in (it might make the argumentation more easily understabdable if you identify what the universal set is a set of though), and determine the correct subsets which describes the varius sub- ject matters of the argument.



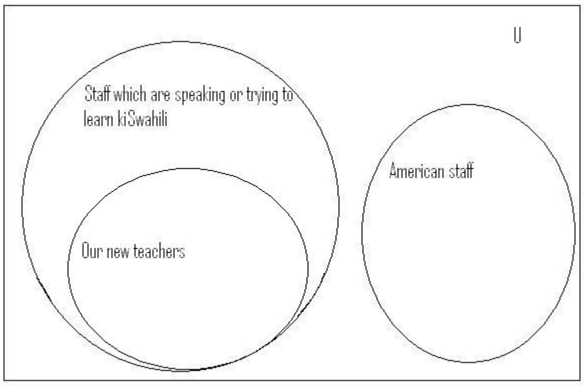
The undescribed universal set that everything we talk about will be part of

We might then add a statement such as "None of the American staff are trying to learn kiSwahili"



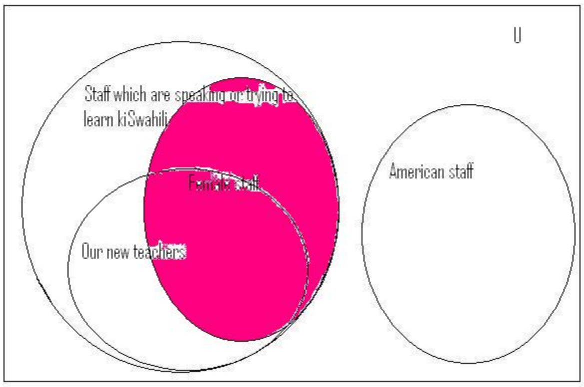
Note that we have to add the set of staff speaking or trying to learn kiSwahilli to show that the intersection between it and the set of American staff is empty.

Now, we can observe that "All our new teachers speak kiSwahili or are learn- ing it"



This way, we can very clearly see that none of our new teachers are American.

We can not, however, know if they're Tanzanian, Danish, Chinese or anything else. If we add another statement such as "All female staff speak or are learning kiSwahili",:



The set of female staff is shown in pink.

we can learn that none of the American staff are female, but we do not know if all our new teachers are female, or if we have teachers that are not new and who don't speak kiSwahili. We need more information to say anything about that.

3 Module 3: Proof by Induction

3.1 Set Theoretic Background for Proofs by Induction

We did start out the sections on Set THeory by declaring that a set is completely determined by its ele- ments. Mathematical induction is a principle of proving statements about natural numbers using this fact.

The natural numbers or the counting numbers is the set $\{1, 2, 3, 4, 5, \dots\}$, where we take the dots to mean that we continue in the same manner. The natural numbers is an infinite set, so we can't prove anything about them by proving it for one element at a time, we would never finish.

What we do is to observe certain things about the nature of the elements of this set:

1) We start with 1, and then we get each new element by adding 1.

2) We can immediately compare any 2 elements to each other and determine which is greatest and which is smallest. If a set has this characteristic, we call it completely ordered.

Note that any set with these 2 characteristics must be more or less identical to the natural numbers. As it is the elements of the set of natural numbers that give it these characteristics, we see how a set and its nature is determined by its elements.

3.2 Proof by Induction

Now, to prove a statement about the natural numbers, we use these characteristics as follows:

We start by checking that the statement is true for 1 (or 2 or some other small number), this is called the **START** of the induction.
We then assume that statement is true some natural number n or $n-1$. This is called the **INDUCTION STEP** or **INDUCTION HYPOTHESIS**.

Finally, we prove that if it holds for n, then it will also hold for $n + 1$ (alternatively, if it holds for $n - 1$, then it also holds for n). This is the **PROOF BY INDUCTION**.

This way of proving a statement is valid because you can get any new natural number by adding 1 to the last one you had, and we have shown that if the statement holds for any natural number n, then it also holds for $n + 1$.

Here's an example of a proof by induction:

We wish to prove the identity $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

START: Check that the identity holds for very small values of n.

$n = 1 : 1 = \frac{1\times(1+1)}{2} = \frac{2}{2} = 1$
$n = 2 : 1 + 2 = 3 = \frac{2(2+1)}{2} = \frac{6}{2} = 3$

INDUCTION HYPOTHESIS: Since the identity holds for small values of n, we assume that it holds for all natural numbers k such that $k \leq m$, where m denotes some arbitrary but set natural number.

CONCLUSION: As we have assumed that the identity holds for all $k \leq m$, we need to check what happens for $m + 1$:

$\sum_{i=1}^{m+1} i =$
$\sum_{i=1}^{m} i + (m+1) =$ (Splitting the sum)
$\frac{m(m+1)}{2} + (m+1) =$ (Using the induction hypothesis)
$\frac{m^2+m}{2} + \frac{2m+2}{2} =$
$\frac{m^2+3m+2}{2} =$
$\frac{(m+1)(m+2)}{2}$

which is exactly what we wanted to prove.

The other way of proving something by using mathematical induction is really the same, except the induction step is slightly different. Instead of assuming the claim to hold for some set, but arbitrary m, you assume that it holds for all the elements of the set $\{1, 2, 3, \ldots, n\}$, and then you show that if that is true, then the claim also holds for $n + 1$.

The trick is to find a way of splitting the sum, product, difference or whatever you are trying to prove in a way so that you can use the induction hypothesis.

4 Module 4: Quantifiers

4.1 The Existence Quantifier

Recall how in a previous module the proposition "All cats in Tanzania are grey" was negated by the existence of just one cat in Tanzania that isn't grey. This concept of the one case that breaks the rule (or less commonly confirms it) is so important in mathematics that it has its own symbol:

The existence quantifier: $\exists$. It is read "there exists".

So if I want to write about the cat which is not grey, I can write: $\exists$ a cat in Tanzania such that this cat is not grey.

Note that we replace "such that" or "so that" with a colon or a vertical line. Thus the statement becomes:

$\exists$ a cat in Tanzania | this cat is not grey.

or

$\exists$ a cat in Tanzania: this cat is not grey.

Note that this does not mean that we can't have 500 cats which are not grey. As long as there is **at least one**, the statement is true.

The existence quantifier states that at least one part of the quantity it describes has the property that comes after the colon or vertical line.

4.2 The Universal Quantifier

If I don't want to negate my statement about the grey cats, I can use the universal quantifier: $\forall$. It is read "for all". The statement "All cats in tanzania are grey" then becomes:

$\forall$ cats in Tanzania | this cat is grey.

Note that the vertical line (or the colon) this time rather means something like "it is true that". If we can find just case that does not have the described quality (here: being grey), then the proposition is false.

The universal quantifier equips each and every part of the quantity it describes with the property that comes after the colon or vertical line.

4.3 Negating the Existence Quantifier

What is the opposite of the existence of at least one case of the described quality?

It is that none of the parts of the quantity in question has the described quality, or that it is true for all of them that they do not have the described quality.

In other words: Negating "$\exists x \in A|x$ has quality a" yields: "$\forall x \in A|x$ does not have quality a".

4.4 Negating the Universal Quantifier

Similarly, if we negate the universal quantifier, we get the existence quantifier:

"$\neg\forall$ cats in Tanzania | this cat is grey" is "$\exists$" a cat in Tanzania | this cat is not grey.

4.5 Negating Composite Statements with Quantifiers

When we need to negate propositions with more than one quantifier, we do it successively.

If we want to negate the composite statement that in words reads: "For all types of bananas that grow in Tanzania it is true that there exists people who will eat them while they're still green", we need to put it in more mathematical terms first:

$\forall$ types of bananas that grow in Tanzania $|\exists$ people in Tanzania | this person eats these bananas green.

Now we have to break it down to statements:

$$\underbrace{\forall \text{ types of bananas that grow in Tanzania}}_{\text{statement 1}} | \underbrace{\exists \text{ a person in Tanzania}|\text{this person eats these bananas green}}_{\text{statement 2}}.$$

Note that we don't call them propositions as no truth value can be determined for a statement like: "For all types of bananas in Tanzania".

First, we negate statement 1: $\underbrace{\forall \text{ types of bananas that grow in Tanzania}}_{\text{statement 1}}$.

As explained in the previous section, we negate $\forall$ by replacing it with $\exists$, so we get:

$$\underbrace{\exists \text{ a type of banana that grows in Tanzania}}_{\neg \text{ statement 1}}$$

Then we look at statement 2:

$$\underbrace{\exists \text{ a person in Tanzania}|\text{this person eats these bananas green}}_{\text{statement 2}}.$$

We notice that statement 2 is itself composite:

$$\underbrace{\exists \text{ a person in Tanzania}}_{\text{statement 2a}}|\underbrace{\text{this person eats these bananas green}}_{\text{statement 2b}}$$

Negating statement 2a, we get:

$$\underbrace{\forall \text{ people in Tanzania}}_{\neg \text{ statement 2a}}$$

And finally, negating statement 2b, we get:

$$\underbrace{\text{this person does not eat these bananas green}}_{\neg \text{ statement 2b}}$$

So the negation of statement 2 becomes:

$$\underbrace{\forall \text{ people in Tanzania}|\text{ this person does not eat these bananas green}}_{\neg \text{ statement 2}}$$

Now, we put everything back together to get:

$$\underbrace{\exists \text{ a type of banana that grows in Tanzania}}_{\neg \text{ statement 1}}|\underbrace{\forall \text{ people in Tanzania}|\text{this person does not eat these bananas green}}_{\neg \text{ statement 2}}$$

To put it in human words, we have that there exists a type of bananas in Tanzania that nobody east while they're green.

Note that it is always a good idea to check if the statement you have negated makes sense in plain words. This is not always easy, but it can enable you to catch errors. Similarly, you can check if your arguments against something make sense by breaking them into mathematical statements and negating them.

4.6 Negating Various Mathematic Symbols

Of course we can negate other mathematical symbols than ∃ and ∀. Here's a table of mathematical symbols and what they are when you negate them:

Symbol	$\neg$ Symbol
$\in$	$\notin$
$\geq$	$<$
$\leq$	$>$
$=$	$\neq$
$\vee$	$\wedge$
$\wedge$	$\vee$
$\subseteq$	$\not\subseteq$
$\exists$	$\forall$
$\forall$	$\exists$

Note that for some symbols, you need to take a longer route because you can't just negate them. Here's a table of those that we have met this far:

Expression	Change to:	Negate that:
$p \rightarrow q$	$(\neg p \vee q)$	$(p \wedge \neg q)$
$p \leftrightarrow q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$	$(\neg p \vee \neg q) \wedge (p \vee q)$

Note that when you negate an expression such as $(p \vee q)$, you need to negate all the terms of it to get $(\neg p \wedge \neg q)$.

5 Module 5: Truth tables

5.1 Propositions

A proposition as a statement that can be either true or false. If no truth value can be determined for some sentence, then it is not a proposition. Here are some examples of propositions:

"My math teacher is a lady."

"Some bananas can be cooked while they are green."

"People can drink water."

"All cats in Tanzania are grey."

"This sentence is written in kiSwahili."

The frist proposition is true if the math teacher is a lady, and false if the math teacher is a man. The second and third propositions are obviously true as we cook green bananas and drink water, while the fourth proposition will be proven wrong by the first cat we see that isn't grey. The last proposition is obviously false as it is written in English.

Here are some examples of sentences which are not propositions. Try assigning truth values for them yourselves

"It will snow in Dar es Salaam 3 months from today."

"x is an integer."

"If this sentence is true, then this sentence is false, but if it is false, then it is true."

Note that the first sentence is verifiable: If it snows in Dar 3 months from today, then it is true, but if it doesn't snow in Dar 3 months from today, then it is false. However, we can't know what will happen 3 months from today, however unlikely it is that it will ever snow in Dar es Salaam.

Note that propositions are sometimes called statements or claims.

5.2 Logical And

The logic "and" and the logic "or" differs somewhat (but not a lot) from the way we use "and" and "or" in our everyday conversations. Let's start with the logic "and":

Given 2 propositions **proposition 1** and **proposition 2**, we say that the composite proposition **proposition 1 *and* proposition 2** is true if both of them are true. To look at our examples from above, if we take 2 true propositions:

"Some bananas can be cooked while they are green."

"People can drink water."

We can make the composite proposition:

"Some bananas can be cooked while they are green **and** people can drink water."

which is obviously still true. However, if we make a composite proposition in which one of the original propositions is false, then the composite proposition is false. Hence both of the following propositions are false:

"People can drink water **and** all cats in Tanzania are grey."

"This sentence is written in kiSwahili **and** some bananas can be cooked while they are green."

If our 2 propositions are called p and q, then we write "p and q" as $p \wedge q$.

Note that you can make composite propositions with more than just 2 propositions.

5.3 Logical Or

We can make composite propositions using logical or just as we made them using logical and. The difference is that with a logical or, it is enough for one of the original propositions to be true. The composite proposition will also be true if both of them are true. It will only be false if both of them are false. The following 3 composite propositions are true:

"Some bananas can be cooked while they are green **or** people can drink water."

"People can drink water **or** all cats in Tanzania are grey."

"This sentence is written in kiSwahili **or** some bananas can be cooked while they are green."

The following composite proposition is false:

"All cats in Tanzania are grey **or** this sentence is written in kiSwahili."

If our 2 propositions are called p and q, then we write "p or q" as $p \vee q$.

5.4 Logical Not

Just as we can negate the meaning of a sentence in English, we can negate the truth value of a proposition. Consider the following:

"All cats in Tanzania are grey."

"All cats in Tanzania are **not** grey."

"Some bananas can be cooked while they are green."

"**No** bananas can be cooked while they are green."

Note that the way we negate a spoken language sentence differs a bit with the structure of the sentence. To negate a proposition "p", we write "¬ p". If you negate a true proposition, it becomes false, and if you negate a false proposition, it becomes true.

5.5 Truth Tables

A truth table is a table that displays the truth value of a proposition. If we have only one proposition, say p, then the truth table for p and $\neg p$ will look like this:

p	$\neg p$
True	False
False	True

If we try adding another proposition, say q and look at $p \wedge q$ (p and q) and $p \vee q$ (p or q), the truth table becomes more complicated:

p	q	$p \wedge q$	$p \vee q$
True	True	True	True
True	False	False	True
False	True	False	True
False	False	False	False

6 Module 6: The Logical Calculus

6.1 Implications

If we think about everyday language, we may argue that A leads to B, B leads to C, and C leads to D, or that A implies B, B implies C, and C implies D. In math, nearly all argumentation follows the same pattern. Given 2 propositions p and q, we write $p \to q$ to say that p implies q. We call p the hypothesis and q the conclusion.

The truth table for $p \to q$ is as follows:

p	q	$p \to q$
True	True	True
True	False	False
False	True	True
False	False	True

We may not have expected that a false hypothesis implies anything and everything, but it will (hopefully) come to make sense later. Note that we say that p implies q is the truth value of $p \to q$ is true.

Here are some examples of true implications:

"Some bananas can be cooked while they are green $\rightarrow$ people can drink water."

"This sentence is written in kiSwahili $\rightarrow$ some bananas can be cooked while they are green."

"All cats in Tanzania are grey $\rightarrow$ this sentence is written in kiSwahili"

Note that though the sentences we make up this way do not make much sense in English, it still makes sense to speak of whether the implications are true or false. All 3 nonsensical sentences above are true implications. However, the following implication is false:

"People can drink water $\rightarrow$ all cats in Tanzania are grey".

In short, one may remember the truth table by the following rule: **Something true can only imply something true, but something false can imply anything.**

6.2 Bi-implications

The bi-implication or bi-conditional (also known as "if and only if") is the same as saying that both $p \rightarrow q$ and $q \rightarrow p$ are true, or that whenever you know one of them to be true, the other is also true. The truth table for $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ is shown below:

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \leftrightarrow q$
True	True	True	True	**True**	**True**
True	False	False	True	**False**	**False**
False	True	True	False	**False**	**False**
False	False	True	True	**True**	**True**

6.3 Tautologies

A tautology is a composite proposition for which all entries in the truth table are true. Similarly, a contradiction is a composite proposition for which all entries in the truth table are false.

Here are the truth tables for a few of the tautologies on page 16 of the chapter on logic:

p	$\neg p$	$p \vee \neg p$
True	False	**True**
False	True	**True**

p	$\neg p$	$(p \wedge \neg p)$	$\neg(p \wedge \neg p)$
True	False	False	**True**
False	True	False	**True**

p	q	$(p \vee q)$	$p \to q$
True	True	True	**True**
True	False	True	**True**
False	True	True	**True**
False	False	False	**True**

And finally the truth table for a contradiction:

p	$\neg p$	$p \wedge \neg p$
True	False	**False**
False	True	**False**

Remember where to find the list of tautologies for future reference. They will form the basis of how we argue later on, and thus it is very important to understand them and to know where to find them.

6.4 Logical Implication

We say that p logically implies q and write $p \Rightarrow q$ if the implication $p \to q$ is a tautology.

Recall that p and q are arbitrary propositions. We always just talk about the truth value of them, not the actual content. Here is one example of a logical implication:

p	q	$(p \vee q)$	$p \to (p \vee q)$
True	False	True	**True**
False	True	True	**True**
True	False	True	**True**
False	False	False	**True**

Thus, we can write $p \Rightarrow (p \vee q)$.

6.5 Logical Equivalence

Similarly, we say that p and q are logically equivalent if $p \leftrightarrow q$ is a tautology.

Note that we may also argue that 2 propositions are logically equivalent by showing that they have the same truth tables. Recall the truth table for $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$:

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \leftrightarrow q$
True	True	True	True	**True**	**True**
True	False	False	True	**False**	**False**
False	True	True	False	**False**	**False**
False	False	True	True	**True**	**True**

This shows that whenever the original propositions p and q have the same truth values, then $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ have the same truth values. and this the 2 composite propositions $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ are logically equivalent and we may write $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$.

Note that when 2 propositions are logically equivalent, we can substitute one for the other and choose the one we like best.

6.6 The Logical Calculus

In order to use the logical calculus, we need to be rather familiar with the list of tautologies here below. In the table, **t** means any tautology (a composite proposition which is always true irregardless of the combination of truth values of the original propositions) and **c** means any contradiction (a proposition which is always false irregardless of the combination of truth values of the original propositions).

Note that 4. - 16. are logical equivalences while 17. - 25. are logical implications.

Number	Tautology	Name
1.	$p \vee \neg p$	
2.	$\neg(p \wedge \neg p)$	
3.	$p \to p$	
4a. 4b.	$p \leftrightarrow (p \vee p)$ $p \leftrightarrow (p \wedge p)$	idempotent laws
5.	$\neg\neg p \leftrightarrow p$	double negation
6a. 6b. 6c.	$(p \vee q) \leftrightarrow (q \vee p)$ $(p \wedge q) \leftrightarrow (q \wedge p)$ $(p \leftrightarrow q) \leftrightarrow (q \leftrightarrow p)$	commutative laws
7a. 7b.	$(p \vee (q \vee r)) \leftrightarrow ((p \vee q) \vee r)$ $(p \wedge (q \wedge r)) \leftrightarrow ((p \wedge q) \wedge r)$	associative laws
8a. 8b.	$(p \wedge (q \vee r)) \leftrightarrow ((p \wedge q) \vee (p \wedge r))$ $(p \vee (q \wedge r)) \leftrightarrow ((p \vee q) \wedge (p \vee r))$	distributive laws
9a. 9b. 9c. 9d.	$(p \vee \mathbf{c}) \leftrightarrow p$ $(p \wedge \mathbf{c}) \leftrightarrow \mathbf{c}$ $(p \vee \mathbf{t}) \leftrightarrow \mathbf{t}$ $(p \wedge \mathbf{t}) \leftrightarrow p$	identity laws
10a. 10b.	$\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$ $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$	De Morgan's Laws
11a. 11b. 11c.	$(p \leftrightarrow q) \leftrightarrow ((p \to q) \wedge (q \to p))$ $(p \leftrightarrow q) \leftrightarrow ((p \wedge q) \vee (\neg p \wedge \neg q))$ $(p \leftrightarrow q) \leftrightarrow (\neg p \leftrightarrow \neg q)$	equivalence
12a. 12b.	$(p \to q) \leftrightarrow (\neg p \vee q)$ $\neg(p \to q) \leftrightarrow (p \wedge \neg q)$	implication
13.	$(p \to q) \leftrightarrow (\neg q \to \neg p)$	contrapositive
14.	$(p \to q) \leftrightarrow ((p \wedge \neg q) \to \mathbf{c})$	reductio ad absurdum
15a. 15b.	$((p \to r) \wedge (q \to r)) \leftrightarrow ((p \vee q) \to r)$ $((p \to q) \wedge (p \to r)) \leftrightarrow (p \to (q \wedge r))$	
16.	$((p \wedge q) \to r) \leftrightarrow (p \to (q \to r))$	exportation law
17.	$p \to (p \vee q)$	addition
18.	$(p \wedge q) \to p$	simplification
19.	$(p \wedge (p \to q)) \to q$	modus ponens
20.	$((p \to q) \wedge \neg q) \to \neg p$	modus tollens
21.	$((p \to q) \wedge (q \to r)) \to (p \to r)$	hypothetical syllogism
22.	$((p \vee q) \wedge \neg p) \to q$	disjunctive syllogism
23.	$(p \to \mathbf{c}) \to \neg p$	absurdity
24.	$((p \to q) \wedge (r \to s)) \to ((p \vee r) \to (q \vee s))$	
25.	$(p \to q) \to ((p \vee r) \to (q \vee r))$	

Basically, the logical calculus is about recognising collections of expressions and substituting them for the expressions they're leading to. In order to do this, we must of course be able to find these collections of expressions and then know what we can replace them with.

It is customary to list your expressions vertically. Let's take an easy example first. Given the expressions p and $(p \to q)$, we get:

1)	p	hypothesis
2)	$(p \to q)$	hypothesis
3)		

We recognise that $p \land (p \to q) \to q$ is a tautology, so we can replace the 2 original expressions p and $(p \to q)$ with q. Normally, we write this as:

1)	p	hypohesis
2)	$(p \to q)$	hypothesis
3)	q	modus ponens of 1) and 2)

And we read it as "p and $(p \to q)$ leads to q" or "q is a logical consequence of p and $(p \to q)$". Note that this really means that we assume that p and $(p \to q)$ are true.

To take a slightly longer example, suppose that we have that:

1)	$(p \land q) \lor (\neg p \land \neg q)$	hypothesis
2)	q	hypothesis
3)		

We recognise that $(p \land q) \lor (\neg p \land \neg q)$ is equivalent to $p \leftrightarrow q$, and that $p \leftrightarrow q$ implies $q \to p$, so we add it before the conclusion:

1)	$(p \land q) \lor (\neg p \land \neg q)$	hypothesis
2)	q	hypothesis
3)	$q \leftrightarrow p$	logical equivalence of 1)
4)	$q \to p$	logical consequence of 3)
5)		

Now we know that $q \land (q \to p) \to p$, so we get:

1)	$(p \land q) \lor (\neg p \land \neg q)$	hypothesis
2)	q	hypothesis
3)	$q \leftrightarrow p$	logical equivalence of 1)
4)	$q \to p$	logical consequence of 3)
5)	p	modus ponens of 2) and 4)

The logical calculus helps us reduce a large amount of propositions to something that's intelligible. With practise, you'll also be able to recognise the pattern of argumentations outlined in the logical calculus whenever you read a mathematical proof.

7 Module 7: Methods of Proof

7.1 Examples

To give an example, recall that the integers, $\mathbb{Z}$, is the set $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$, and that the natural numbers, $\mathbb{N}$, is the set $\{x \in \mathbb{Z} | x > 0\} = \{1, 2, 3, 4, \dots\}$.
We want to prove the following:

Theorem 1: $\forall n \in \mathbb{Z} : n^3 > 0 \Leftrightarrow n > 0$

The theorem states that for all integers n, we have that n^3 is greater than 0 if and only if n itself is greater than 0. We'll prove it by breaking it into parts:

Part 1 ($n > 0 \Rightarrow n^3 > 0$):. Assume that $n > 0$. $n^3 = n \cdot n \cdot n$, and as $n > 0$, we haven't multiplied by anything negative, so n^3 must be greater than 0.

This is a direct proof ($p \Rightarrow q$).

Part 2 ($n^3 > 0 \Rightarrow n > 0$): Assume that $n < 0$. As $n^3 = n \cdot n^2$ and we know that for all integers m, $m^2 > 0$, we have that $n \cdot n^2 < 0$ as we assumed that $n < 0$ and something negative multiplied by something positive yields a negative number. So $n^3 < 0$.

This is a proof by contraposition ($(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$).

Theorem 2: $\forall n, m \in \mathbb{N} : n, m$ are even, $n + m$ is even.

DIRECT PROOF: This is $p \Rightarrow q$. Assume that n, m are even. Then we can find s, t so that $n = 2s$ and $m = 2t$. Now, $n + m = 2s + 2t = 2(s + t)$. As s, t are integers, $(s + t)$ is an integer, say $s + t = k$, and thus $n + m = 2(s + t) = 2k$, which is an even integer.

INDIRECT PROOF / PROOF BY CONTRAPOSITION: This is $\neg q \Rightarrow \neg p$. Assume that $n + m$ is odd. Now, we can find j so that $n + m = 2j + 1$, or $n = 2j - m + 1$. If we assume that either n or m is odd, we're done with the proof, so assume both of them are even. Now, we can find s, t so that $n = 2s$ and $m = 2t$. Inserting this in the formula, we get $2s = 2j - 2t + 1 = 2s = 2(j - t) + 1$. As j, t are integers, $(j - t) = u$ is an integer, and thus we have that $2s = 2u + 1$. This is clearly impossible, so either n or m must be odd.

Note that the indirect proof required the use of another logical equivalence, $(x \rightarrow y) \leftrightarrow (x \wedge \neg y) \rightarrow c$, which is also used below:

PROOF BY REDUCTIO AD ABSURDUM This is $(p \wedge \neg q) \Rightarrow c$. Assume that n, m are even and that $n+m$ is odd. We can find s, t, u so that $n = 2s, m + 2t$ and $n + m = 2u + 1$. Now, we have that $n + m = 2s + 2t = 2(s + t) = 2u + 1$, which is clearly impossible as $s + t$ is an integer.

There's a bit more to be said about proof by substitution. We may look at the table of tautologies to find any form r that will suit our purposes. This takes some practise, though.

7.2 The Most Common Methods of Proof

Here is a table of things to prove and methods to prove them. Recall from previous modules that a contradiction is a (composite) proposition that is always false no matter what the original propositions were. In the table below, c denotes a contradiction:

Proposition	Method 1	Method 2	Method 3	Method 4
$p \Rightarrow q$ Implication	$p \Rightarrow q$ Direct Proof	$\neg q \Rightarrow \neg p$ Contraposition	$(p \wedge \neg q) \Rightarrow c$ Reductio ad Absurdum	$(p \Rightarrow r) \wedge (r \Rightarrow q)$ Proof by Substitution
$p \Leftrightarrow q$ Bi-implication	$p \Leftrightarrow q$ Direct Proof	$\neg p \Leftrightarrow \neg q$ Contrapositive	$(p \Rightarrow q) \wedge (q \Rightarrow p)$ Breaking into Parts	$(r \Leftrightarrow q) \wedge (p \Leftrightarrow r)$ Proof by Substitution

Note that if we choose the proof by breaking into parts method, we can go to the first line and choose any of the methods of proof for a single implication, and we can even prove the 2 implications in 2 different ways.

8 Module 8: Language and Logic

8.1 The Link between Language and Logic

This is a section that may not be present in all textbooks. It is relevant because it is when we can see the link between language and logic that the demystification of the subject matter and the deeper understanding occurs, and also it seems to be a very challenging subject matter for most students in spite (or maybe because) of the fact that it is revealing the link between how we speak and argue every day and the mathematical formulas and symbols.

When we argue, we tend to sum up a number of premises, draw logical conclusions from them and present the new set of premises, just as in mathematical proofs. If our opponents can accept the premises and the logic that we have used, they're forced to accept the final conclusions as well. Of course, in the end of the day, a lot of every day arguing comes down to personal beliefs, which can not be quantified, but the rethorical skill of arguing logically is never the less very useful in many situations.

On the other hand side, with the support of everyday language it becomes easier to catch and correct our own mistakes when we deal with mathematical logic. A simple example is the following: Let P be the set of all people, and let $s(x,y)$ mean that x and y are siblings. Then $\forall x \in P : \exists y \in P : s(x,y)$ means that all people have a sister or a brother, while $\exists x \in P : \forall y \in P : s(x,y)$ means that there exists a person who's the brother or sister of every other living person on this planet. While the first statement is wrong because we know of single children, the last statement would imply that all living people were born form the same mother and father!

8.2 What is a Mathematical Proposition?

The fundamental skill required is being able to translating between everyday lanuage and symbolic expressions.

A mathematical proposition is a statement that can either be true or false. Some examples could be:

"Bananas are green before they turn yellow."

"ePhoney is the cheapest internet based telephony software."

"There are no female math professors in Denmark."

In short, if you can't answer it with a yes or a no, then it is not a mathematical proposition. Some examples of sentences that are not propositions:

"x is an integer."

"It will rain tomorrow."

"People who like abstract paintings have no taste."

8.3 Connecting Propositions to form Statements

We may form statements by connecting propositions pairwise by connectives, e.g. a colon "**:**" is read as "such that" or "so that", an implication arrow $p \rightarrow q$ is aread as "p implies q" or "If p then q", etc. One example is:

Proposition p: The wind is from Kilimanjaro.
Proposition q: The wind is cold.

We may now connect p and q to form the statement $p \to q$: "If the wind is from Kilimanjaro, then it is cold".

We may also start with a quantifier. With p and q from above, we can get a similar statement by using the universal quantifier $\forall$ to get "$\forall$ winds from Kilimanjaro: q" or in plain words "all winds from Kilimanjaro are cold."

If we had used the existance quantifier instead, we'd get the statement: "$\exists$ a wind from Kilimanjaro: q" or in plain words "there exists a wind from Kilimanjaro which is cold."

What is important here is to recognise that the sentences: "If the wind is from Kilimanjaro, then it is cold", "all winds from Kilimanjaro are cold" and "there exists a wind from Kilimanjaro which is cold" have been constructed using the same 2 propositions, p and q.

8.4 Recognising the Shape of an Argument

Recognising the shape of an argument is basically being able to break the argument into its components and then translating them into their symbolic form. One example:

"Letting your basic hygiene slip during a cholera outbreak causes a hightened risk of cathing cholera, so you won't have a hightened risk of catching cholera if you do not let your basic hygiene slip."

breaks down to the propositions:

Proposition p: "You let your basic hygiene slip during a cholera outbreak."
Proposition q: "You have a hightened risk of catching cholera",

And we see that the statement breaks into "$p \to q \leftrightarrow \neg q \to \neg p$."

Note that this exercise takes a bit of practise as there are several examples of words which may translate into different symbols depending on the situation, e.g. "so" may both translate into "$\to$" (implies) and "$\leftrightarrow$" (is equivalent to).

8.5 Negating Statements

We negate the symbolic expressions sequencially, e.g.

$p = \forall x \in \mathbb{Q} : \exists y \in \mathbb{R} : xy \notin \mathbb{Z}$

is broken down to

$$p = \underbrace{\forall x \in \mathbb{Q}}_{p_1} : \underbrace{\exists y \in \mathbb{R}}_{p_2} : \underbrace{xy \notin \mathbb{Z}}_{p_3}$$

Whish is then negated sequencially:

$$\neg p = \neg(\underbrace{\forall x \in \mathbb{Q}}_{p_1} : \underbrace{\exists y \in \mathbb{R}}_{p_2} : \underbrace{xy \notin \mathbb{Z}}_{p_3}) \rightarrow$$
$$\neg p = \underbrace{\exists x \in \mathbb{Q}}_{\neg p_1} : \neg(\underbrace{\exists y \in \mathbb{R}}_{p_2} : \underbrace{xy \notin \mathbb{Z}}_{p_3}) \rightarrow$$
$$\neg p = \underbrace{\exists x \in \mathbb{Q}}_{\neg p_1} : \underbrace{\forall y \in \mathbb{R}}_{\neg p_2} : \neg(\underbrace{xy \notin \mathbb{Z}}_{p_3}) \rightarrow$$
$$\neg p = \underbrace{\exists x \in \mathbb{Q}}_{\neg p_1} : \neg(\underbrace{\forall y \in \mathbb{R}}_{\neg p_2} : \underbrace{xy \in \mathbb{Z}}_{\neg p_3}$$

Now, turning this into everyday language, we'd get

$$p = \underbrace{\text{For all rational numbers } x}_{p_1} \underbrace{\text{we can find a real number } y}_{p_2} \underbrace{\text{such that the product } xy \text{ is not an intege}}_{p_3}$$

Which we also negate sequentially to get:

$$\neg p = \underbrace{\text{There extsts a rational number } x}_{\neg p_1} \underbrace{\text{such that for all real numbers } y}_{\neg p_2} \underbrace{\text{the product } xy \text{ is an integer}}_{\neg p_3}$$

9 Module 9: Limits, Continuity and Differentiability

9.1 Functions

9.1.1 Definition of a Function

A function is a mathematical object that takes an independent variable, performs some predefined operation on it, and returns the value of the dependent variable. An example could be:

$$\underbrace{y}_{\text{dependent variable}} = \underbrace{f(x)}_{\text{function name and independent variable}} = \underbrace{7 \cdot x^2 + 5 \cdot x - 2}_{\text{function body}}$$

As we can see, the *function body* is a recipe telling us how to get the value of the dependent variable for each choice of the independent variable. Note that as long as we haven't made a choice of independent variable, the function $f(x)$ is just a mathematical recipe.

9.1.2 Uniqueness of the Function's Value

For a mathematical recipe to be a function, we demand that it **uniquely defines** the function's value in any point. We cannot have a function such as:

$$f(x) = \begin{cases} x+3 & \text{on Mondays} \\ 5x-2 & \text{on Tuesdays} \\ -x^2 & \text{on Wednesdays} \\ 0 & \text{on Thursdays} \\ \frac{1}{2x} & \text{on Fridays} \end{cases}$$

Or such as the one given in the table below:

f(x)	1	1	3	4	5
x	3	4	5	3	5

Imagine using the first one in the construction of a house. It would be a very strange house indeed, changing its shape each day of the week and being undefined on weekends. I wouldn't want to live there!

The problem with the second is that it assigns 2 different values to the number 1. This violates the principle that a function must uniquely define the value that each choice of independent variable evaluates to.

9.1.3 Domain and Range

The domain of a function is the largest set of numbers that can be evaluated using the recipe given in the function body of that particular function. With a lot of functions, the domain is simply all of the the real numbers, $\mathbb{R}$. A point (a real number) is excluded from the domain of a function in the case that it will cause an illegal (mathematically speaking) action if you try to evaluate the function in that point.

The most common illegal action is division by 0. We may under no circumstances divide by 0. To give an example of this, consider the function:

$f(x) = \frac{x}{x-1}$

If $x = 1$, the denominator becomes $1 - 1 = 0$, so the domain of this function cannot include the number 1. All other reals are fine, so in this case the domain of the function, often denoted $\mathcal{D}(f(x))$, becomes all the real numbers except 1, $\mathcal{D}(f(x)) = \mathbb{R} \setminus \{1\}$.

Note that we may for other purposes chose to define a smaller domain. In that case we must supply a specification of the domain over which we want to evaluate the function.

The range if a function is the set of numbers that the function can evaluate to over its domain. If our function is

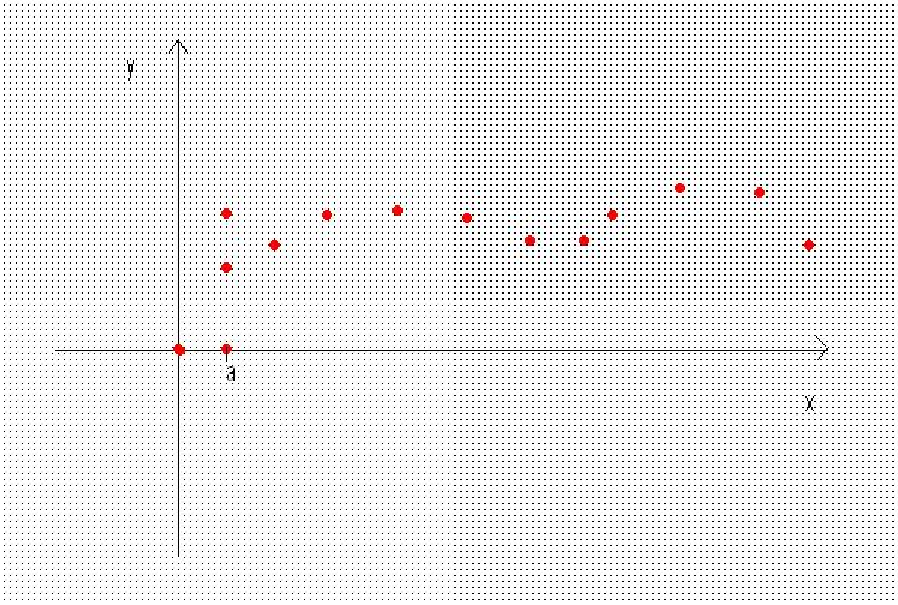
$f(x) = x + 2$

and we have specified the domain as $\mathcal{D}(f(x)) = \{1, 2, 3, 4, 5\}$, then the range of the function is the set $\mathcal{R}(f(x)) = \{3, 4, 5, 6, 7\}$. If we define the domain to be $\mathcal{D}(f(x)) = [1, 5]$ (the interval from 1 to 5), then we get $\mathcal{R}(f(x)) = [3, 7]$ (the interval from 3 to 7).

9.1.4 Graphs

The graph of a function is the graphical representation of that function, drawn in a coordinatesystem such that the points are the independent variable and its functional value, $(x, f(x))$. The graph will always be incomplete except in the cases where we have restricted our domain to a small part of the real line. When the domain hasn't been specified (and thus must be assumed to the all of the real line), we must remember that we look only at a part of the graphical represetation of that function.

We can determine if something is a function by looking at the graph. Given a situation in which any value of the x-axis has more than one second coordinate, we know that this isn't a function:

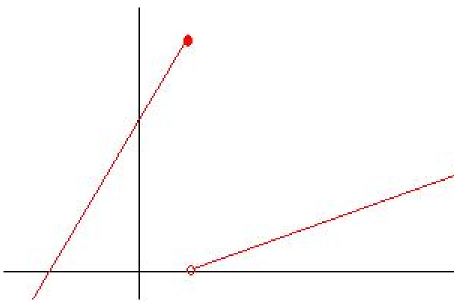


The above points cannot denote a function because there are 3 different second coordinates at the point a.

If we want to illustrate that the function makes a jump, we do as follows. Consider the function:

$$f(x) = \begin{cases} x+3 & x \leq 2 \\ x-2 & x > 2 \end{cases}$$

And we use a closed dot to denote that the point $(2,5)$ is included in the graph, whereas the point $(2,0)$, denoted by an open dot, is not included in the graph:



If both of them had been closed, then the picture could not be of a graph of a function, and if both had been open, we would have a point in which the function isn't defined.

9.1.5 Composite functions

A function can easily be evaluated in a point that is the functional value of another function. If $f(x) = x + 2$ and $g(x) = 3x$, then we get the composite function $f(g(x))$ by inserting $g(x)$ on x's place: $f(g(x)) = (3x) + 2$. Note that $g(f(x)) = 3(x+2) = 3x+6$, so the order in which we call the functions does matter. We do sometimes write $f(g(x))$ as $f \circ g(x)$ or $(f \circ g)(x)$.

9.2 Limits

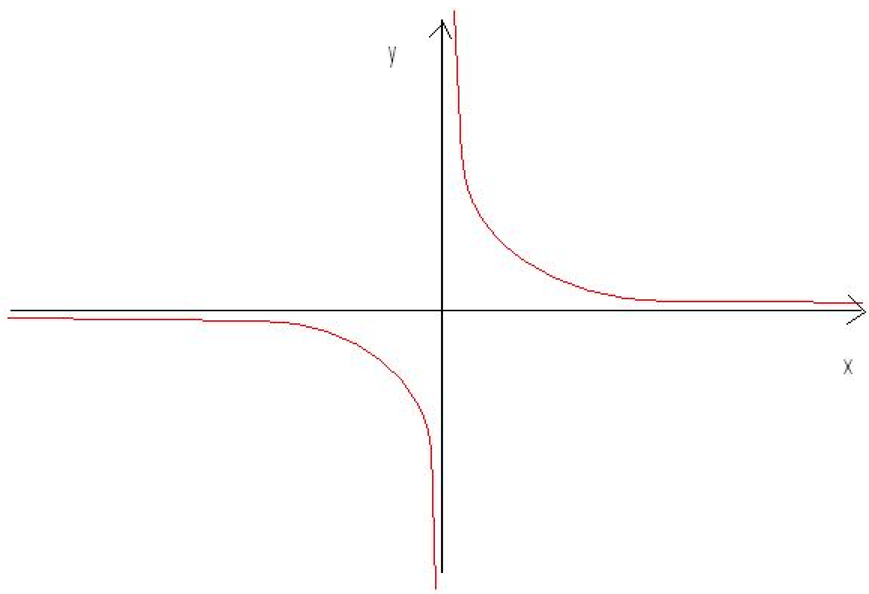
The limit of an expression including some variable means the value, if it exists, that the expression evaluates to as the variable approaches some number. If the expression is called $E(x)$ and we know that $E(x)$ tends to (approaches) L as x tends to a, we write it $lim_{x \to a}(E(x)) = L$.

Some examples are:

$lim_{x \to 2}(x^2) = 4$

$lim_{x \to 3}(x + 2) = 5$

We may exprience situations when the limit doesn't exist. For instance, consider function $f(x) = \frac{1}{x}$. Its graph looks as follows:



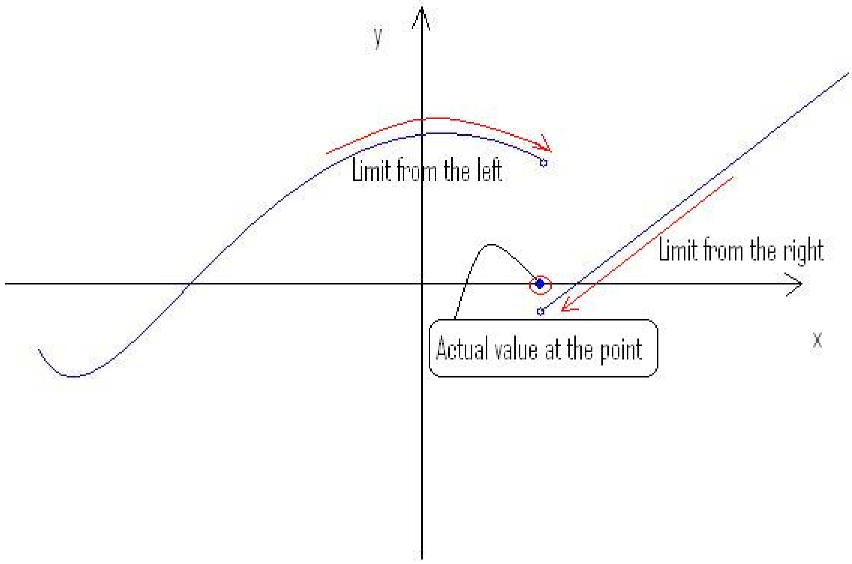
Which value should we assign to $lim_{x \to 0}(f(x))$? Obviously, as x tends to 0 from the left, $f(x)$ tends to $-\infty$ (we may write this $lim_{x \to a^-}(f(x)) = -\infty$), but as x tends to 0 from the right, $f(x)$ tends to ∞ (we may write this

$lim_{x \to a^+}(f(x)) = \infty$). And as $-\infty \neq \infty$, we cannot assign a value to the limit.

In other cases of division by 0, we may be more lucky and find that the expression actually has a limit. Consider the expression $\frac{x}{x-1}$. What happens when x tends to 1? In this case, $lim_{x \to 1}(\frac{x}{x-1})$ exists and equals 1 (using l'Hopital's Rule).

Note that though we can't actually calculate $\frac{x}{x-1}$ for $x = 1$ as this would entail division by 0, we can still find the limit as x tends to 1.

Consider the below function:



It has a jump at the point a. If you look at the red arrows, you can see that $lim_{x \to a^-} f(x) \neq lim_{x \to a^+} f(x)$. Recall that the limit in a is simply the value (y or $f(x)$) that the function approaches as x approaches a. However, consider for a moment if the functional values actually approach $f(a)$ at all. They don't!

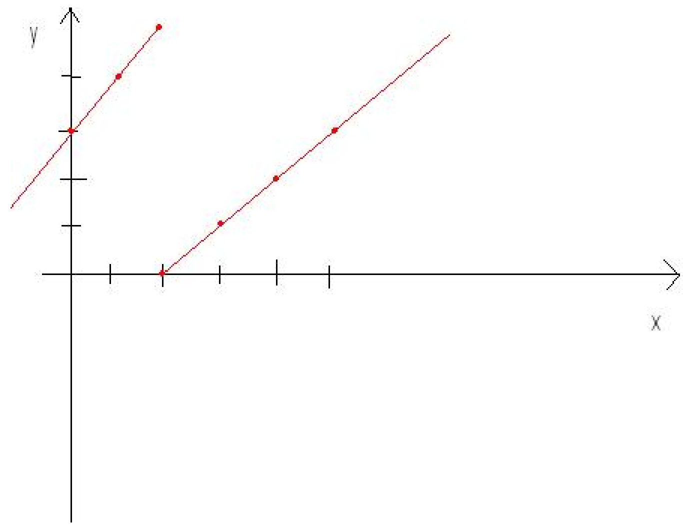
Recall again that the limit in a is the value that the function approaches as x approaches a. If this is not clearly defined, meaning that $lim_{x \to a^-} f(x) = f(x) = lim_{x \to a^+} f(x)$, then we can't assign a value to it.

In the above situation, we have no less than 3 equally qualified candidates for this limit, namely $lim_{x \to a^-} f(x)$, $lim_{x \to a^+} f(x)$ and $f(a)$. As a mathematical quantity can't have 3 different values at the same time, we say that it doesn't exist. It doesn't exist because we can't assign a unique value to it.

9.3 Continuity

Returning to functions, we say that a function $f(x)$ is continuous in the point a if $lim_{x\to a}(f(x)) = L$ and $f(a) = L$ no matter whether we approach a from the left or from the right, or in other words, $f(x)$ is continuous in a if $lim_{x\to a^-}(f(x)) = lim_{x\to a^+}(f(x)) = L$.

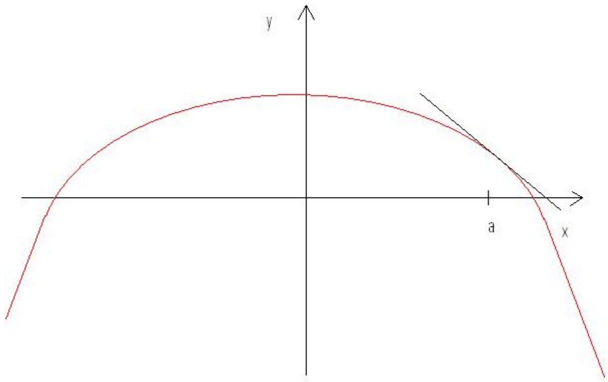
In the case of the real numbers, a continuous function can easily be identified by its graph. If you can draw the graph without lifting your pencil from the paper, then the function is continuous. Thus, our function from earlier:



is discontinuous in the point $x = 2$, but continuous over the intervals $]-\infty, 2]$ and $]2, \infty[$.

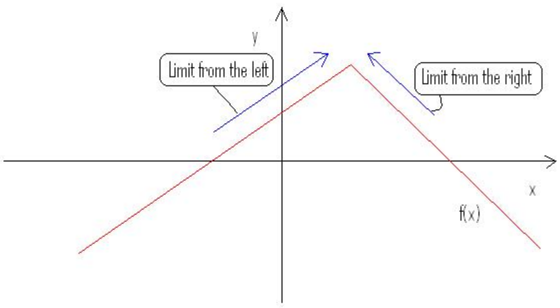
9.4 Derivatives

We can only find the derivative of a function $f(x)$ in the points $x = a$ where $f(x)$ is continuous and doesn't have any "bends". In this case, the derivative of $f(x)$ in the point $x = a$ is the slope of the tangent line of the graph of the function in the point $(a, f(a))$.



This is done by calculating the limit $\lim_{h\to 0}(\frac{f(x+h)-f(x)}{h})$ (refer to the formula for finding the slope of a straight line between 2 points). We write the derivative of $f(x)$ as $\frac{d}{dx}(f(x))$ or $f'(x)$.

Note that the reason why we can't define the derivative in a point where the function has a "bend" is essentially the same as why we can't find a limit in the case where a function has a point of discontinuity. If the function bends, the limit $\lim_{h\to 0^-}(\frac{f(x+h)-f(x)}{h}) \neq \lim_{h\to 0^+}(\frac{f(x+h)-f(x)}{h})$ because these limits are the slope of the function in that point, see picture below.



There are various rules for calculating derivatives. The following table is not exhaustive and the reader is strongly encouraged to look up more rules for calculation on the internet (for example here: http://en.wikipedia.org/wiki/List_-of_differentiation_identities) or in other textbooks. The letter k denotes some constant:

$f(x)$	$f'(x)$
k	0
x	1
$k \cdot x$	k
x^n	$n \cdot x^{n-1}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\frac{1}{x}$	$\ln(\|x\|)$
e^x	e^x
e^{kx}	ke^{kx}
$\cos(x)$	$-\sin(x)$
$\sin(x)$	$\cos(x)$

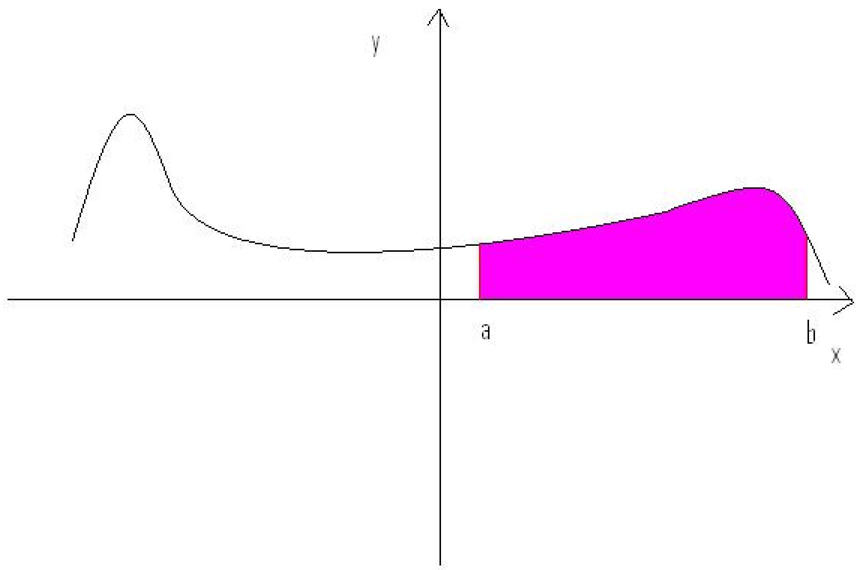
The following rules tells us how to find derivatives of composite functions. Let $g(x)$ denote some other continuous function:

$kf(x)$	$kf'(x)$	(multiplication by a constant)
$f(x) + g(x)$	$f'(x) + g'(x)$	(sum rule)
$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$	(product rule)
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$	(fraction rule)
$f(g(x))'$	$f'(g(x))g'(x)$	(chain rule)

9.5 Integration

Integration is the opposite of derivation. If for some continuous function $F(x)$ we have that $F'(x) = f(x)$, then we also have that $\int f(x)dx = F(x)$. This is also called the indefinite integral of $f(x)$. It is indefinite because the result is another function.

If we have a continuous function, what we do graphically when we integrate, is to calculate the area under the graph of the function:



We say that this is the definite integral of $f(x)$ from a to b, and we write it $\int_a^b f(x)dx$. It is definite because the result is a number. We have that $\int_a^b f(x)dx = F(b) - F(a)$.

The reader is strongly encouraged to look up the rules of integration on the internet (for example here: http://en.wikipedia.org/wiki/Lists_of_integrals) or in another textbook. The following table contains a few of them:

$f(x)$	$F(x)$
k	kx
x	$\frac{1}{2}x^2$
$k \cdot x$	$\frac{k}{2}x^2$
x^n	$\frac{1}{n+1}x^{n+1}$
e^x	e^x

And here are a few of the rules on how to calculate integrals:

1) $\int kf(x)dx = k\int f(x)dx$

2) $\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx$

3) $\int f(x)g(x)dx = F(x)g(x) - \int F(x)g'(x)dx$

4) $\int f(g(x))g'(x)dx = \int f(u)du$, $u = g(x)$